CHL dyons and statistical entropy function from D1-D5 system

Justin R. David and Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, India
E-mail: `justin,sen@mri.ernet.in`, `ashoke.sen@cern.ch`

ABSTRACT: We give a proof of the recently proposed formula for the dyon spectrum in CHL string theories by mapping it to a configuration of D1 and D5-branes and Kaluza-Klein monopole. We also give a prescription for computing the degeneracy as a systematic expansion in inverse powers of charges. The computation can be formulated as a problem of extremizing a duality invariant statistical entropy function whose value at the extremum gives the logarithm of the degeneracy. During this analysis we also determine the locations of the zeroes and poles of the Siegel modular forms whose inverse give the dyon partition function in the CHL models.

KEYWORDS: D-branes, String Duality, Black Holes in String Theory.

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Contents

1. Introduction

There exists a proposal for the exact degeneracy of dyons in toroidally compactified heterotic string theory $[1-5]$ and also in toroidally compactified type II string theory [6] in four dimensions. These formulæ are invariant under the S-duality transformations of the theory, and also reproduce the entropy of a dyonic black hole in the limit of large charges [2]. Ref. [3] proposed a proof of this formula by first relating this to a five dimensional D1-D5 system and then counting the degeneracy of this system.

In $[7-9]$ this conjecture was generalized to a class of CHL models $[10-15]$, obtained by modding out heterotic string theory on $T^2 \times T^4$ by a $\mathbb{Z}_N$ transformation that involves $1/N$ unit of translation along one of the circles of T^2 and a non-trivial action on the internal conformal field theory (CFT) describing heterotic string compactification on T^4. The values of N considered in [7] were $N = $1,2,3,5,7, with $N = 1$ representing toroidal compactification. The conjectured formulæ for the dyon spectrum are invariant under

the duality symmetries of the theory, and also reproduce the black hole entropy for large charges up to first non-leading order.

The goal of this paper is twofold. First we shall give a 'proof' of the conjectured formula by relating the dyons in the four dimensional CHL models to a D1-D5 system in five dimensional CHL models. Although the basic idea behind our analysis is similar to that of [3], the details are different and even for the $N = 1$ case our proof does not reduce exactly to that given in [3] (see [9] for an attempt to generalize the analysis of [3] to CHL models). The other goal will be to develop a systematic procedure for extracting the asymptotic behaviour of the degeneracy formula as a series in inverse powers of charges and non-perturbative corrections. During this analysis we find that the statistical entropy, defined as the logarithm of the degeneracy, is obtained by extremizing a function. We call this function the statistical entropy function in analogy with the black hole entropy function whose extremization gives the entropy of an extremal black hole. The analogy in fact goes further since we show that up to first non-leading order the statistical entropy function of the dyons matches the entropy function of extremal black holes carrying the same charges.

Since the analysis involves a lot of technical details, we have organised the paper such that only the basic ideas are presented in the text and all technical details are relegated to the appendices. In § 2 we describe how the dyons in four dimensional CHL models can be related to a D1-D5 system in five dimensional CHL models. In § 3 we count the degeneracy of the D1-D5 system under consideration and show that the result of this analysis agreees with the conjecture given in [7, 8]. Finally in § 4 we develop a systematic procedure for extracting the asymptotic behaviour of the degeneracy in the limit of large charges.

2. Dyons from D1-D5 system

In this section we shall follow the method of [3] to relate the dyons in a four dimensional CHL model to a rotating D1-D5 system in a five dimensional CHL model. The steps leading to this relation are as follows.

1. Begin with type IIB string theory on $K3 \times S^1$ with Q_5 D5-branes wrapped on $K3 \times S^1$, Q_1 D1-branes wrapped on S^1, $-n$ units of momentum along S^1 and angular momentum J_1 and J_2 in two independent planes with $J_1 + J_2 = J$ [16].[1] Since a K3-wrapped D5-brane carries -1 unit of D1-brane charge, this system has $Q_1 - Q_5$ units of D1-brane charge along S^1. For definiteness we shall choose the coordinate along S^1 such that it has period 2π.[2]

2. Now place this system at the center of Taub-NUT space with coordinates of the Taub-NUT space transverse to $K3 \times S^1$. The orientation of the Taub-NUT space is

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[1]If we identify the tangent space group SO(4) of the four transverse spatial dimensions to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, then J can be identified with twice the $\mathrm{U}(1)_L$ generator of $\mathrm{SU}(2)_L$.

[2]Note that in our convention supersymmetry acts on the right-moving sector of the D1-D5 world-volume theory, i.e. on modes carrying positive momentum along S^1. Thus a BPS state will involve only excitations involving modes with negative or zero momentum along S^1.

chosen such that the isometry direction of the Taub-NUT geometry that becomes the compact direction $\widetilde{S}^1$ in the asymptotic region coincides with the angular coordinate ϕ along which the black hole rotates [17]. The new configuration now corresponds to a state in type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1$ with Q_5 D5-branes wrapped on $K3 \times S^1$, $(Q_1 - Q_5)$ units of D1-brane charge along S^1, $-n$ units of momentum along S^1, a Kaluza-Klein monopole associated with the compact coordinate $\widetilde{S}^1$ and J units of momentum along $\widetilde{S}^1$.

3. Make an S-duality transformation on this system to get type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1$ with Q_5 NS5-branes on $K3 \times S^1$, $(Q_1 - Q_5)$ units of fundamental string winding charge along S^1, $-n$ units of momentum along S^1, J units of momentum along $\widetilde{S}^1$, and a Kaluza-Klein monopole associated with $\widetilde{S}^1$ compactification.

4. Now make an $R \to 1/R$ duality transformation along $\widetilde{S}^1$ to convert the theory into IIA on $K3 \times S^1 \times \hat{S}^1$ with Q_5 Kaluza-Klein monopoles associated with $\hat{S}^1$ compactification, $(Q_1 - Q_5)$ units of fundamental string winding charge along S^1, $-n$ units of momentum along S^1, J units of fundamental string winding charge along $\hat{S}^1$, and a single NS5-brane wrapped on $K3 \times S^1$. Here $\hat{S}^1$ denotes the dual circle of $\widetilde{S}^1$.

5. Finally using the string-string duality [18–22] relating heterotic on T^4 to type IIA string theory on $K3$, we can map this system to heterotic string theory on $T^4 \times S^1 \times \hat{S}^1$ with Q_5 Kaluza-Klein monopoles associated with $\hat{S}^1$ compactification, $(Q_1 - Q_5)$ units of NS5-brane charge along $T^4 \times S^1$, $-n$ units of momentum along S^1, J units of NS5-brane charge along $K3 \times \hat{S}^1$, and a single fundamental string wrapped on S^1. If Q_e and Q_m denote the electric and magnetic charge vectors respectively, then this system has $\frac{1}{2} Q_m^2 = (Q_1 - Q_5)Q_5$, $\frac{1}{2}Q_e^2 = n$, and $Q_e.Q_m = J$.

Now we can mod out both sides of the duality by a $\mathbb{Z}_N$ transformation which, in the original theory, involves $2\pi/N$ translation along S^1 and some order N transformation $\widetilde{g}$ acting on $K3$ that commutes with all the space-time supersymmetry transformations. In the final heterotic string theory the $\mathbb{Z}_N$ transformation acts as $2\pi/N$ translation along S^1 together with some $\mathbb{Z}_N$ action on the coordinates of T^4 and the 16 left-moving internal bosonic coordinates. In order to get a $\mathbb{Z}_N$ invariant configuration so that we can carry out the $\mathbb{Z}_N$ modding, we need to put periodic boundary conditions on all the branes which extend along S^1, and take N identical copies of all the branes transverse to S^1 and place then at intervals of $2\pi/N$ along S^1. The latter set includes the five branes along $K3 \times \hat{S}^1$; we need to take NJ five branes and place them at intervals of $2\pi/N$ along S^1. After orbifolding the direction along S^1 can be regarded as a circle of radius $1/N$, and we shall have J five branes per unit period transverse to S^1. The natural unit of momentum along S^1 is now N, and momentum $-n$ along S^1 can be regarded as $-n/N$ units of momentum. The other charges have the same values as in the parent theory. This gives $\frac{1}{2}Q_e^2 = n/N$. Q_m^2 and $Q_e \cdot Q_m$ remain unchanged from their original values. Thus we have

$$\frac{1}{2}Q_e^2 = n/N, \qquad \frac{1}{2}Q_m^2 = (Q_1 - Q_5)Q_5, \qquad Q_e \cdot Q_m = J. \qquad (2.1)$$

One point to note is that since we have a single fundamental string along S^1, a $\mathbb{Z}_N$ modding would correspond to requiring that under a shift of $2\pi/N$ along S^1 the non-compact coordinates of the string transverse to S^1 come back to their originl values and the coordinates of the string along $K3$ come back to the image of the original location under the $\mathbb{Z}_N$ generator $\widetilde{g}$ acting on K3. This represents a twisted sector state in the orbifold theory. This is consistent with the assertion of [7] that the proposed answer for the degeneracy of dyonic states is valid only for states which carry electric charges compatible with twisted sector states.

This analysis shows that the problem of computing degeneracy of dyons in the four dimensional CHL model can be reduced to the problem of computing the degeneracy of a rotating D1-D5 system in Taub-NUT space, carrying momentum along the direction common to the D1-brane and the D5-brane. We shall now turn to the problem of computing degeneracy of this system.

3. Counting of states of the rotating D1-D5 system in Taub-NUT space

In this section we shall compute the degeneracy of the D1-D5 system described in the previous section. For simplicity we shall restrict our analysis to the $Q_5 = 1$ case, – the generalization of this analysis to the $Q_5 \neq 1$ case has been discussed in appendix D. In this case an intuitive picture of the states of the D1-D5 system may be given as follows [24]. First of all dynamics of the 1-5 strings produces an effective binding between the D1 and D5 which forces the D1-branes to move in the plane of the D5-brane. Thus the only zero modes of the D1-brane associated with the transverse directions are those along K3. A generic state of the system will contain a certain number of isolated D1-branes in the plane of the D5-brane, with the i-th D1-brane wound w_i times along S^1 and carrying momenta $-l_i$ along S^1 and j_i along $\widetilde{S}^1$, with $w_i, l_i, j_i \in \mathbb{Z}$. The BPS condition will also require $w_i > 0$, $l_i \geq 0$. The overall motion of the D1-D5-system in Taub-NUT space can carry some additional momentum $-l_0$ along S^1 and j_0 along $\widetilde{S}^1$. Furthermore the low energy dynamics of closed string modes localized near the core of the Taub-NUT space can also carry some momentum $-l_0'$ along S^1.[3] Thus we have the constraint

$$\sum_i w_i = Q_1, \qquad l_0' + l_0 + \sum_i l_i = n, \qquad j_0 + \sum_i j_i = J. \qquad (3.1)$$

Each of the systems described above must satisfy the boundary condition that under a translation along S^1 by $2\pi/N$ its oscillation modes along $K3$ get twisted by $\widetilde{g}$.

Our goal is to compute the degeneracy of the D-brane system described above. In fact what we shall really compute is not the degeneracy, but an appropriate index defined as follows. First of all the D1-D5-Kaluza Klein monopole system in type IIA on $K3 \times S^1 \times \widetilde{S}^1$ breaks twelve of the sixteen supercharges of the bulk theory. Thus there will be twelve fermionic zero modes associated with the broken supersymmetry generators. Quantization

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[3]Localized low energy excitations around the Taub-NUT space do not carry any momentum along $\widetilde{S}^1$. They can carry fundamental string winding charge along $\widetilde{S}^1$ [23], but for the configuration which is of interest to us, these charges vanish.

of these fermion zero modes produce a $2^6 = 64$-fold degeneracy of states. These states form a single irreducible multiplet of the supersymmetry algebra known as the intermediate multiplet containing equal number of fermionic and bosonic states. This is then tensored with the states obtained by quantizing the rest of the degrees of freedom of the theory to get the full spectrum. If the state that is tensored with the basic supermultiplet is bosonic we shall call this a bosonic supermultiplet and if this state is fermionic we shall call this a fermionic supermultiplet. In other words we shall call a supermultiplet bosonic (fermionic) if the highest spin state in the supermultiplet is bosonic (fermionic). We shall denote by $h(Q_1, n, J)$ the number of bosonic supermultiplets minus the number of fermionic supermultiplets of the system described above carrying quantum numbers (Q_1, n, J). By an abuse of notation we shall continue to refer to this number as the degeneracy of states.

We shall first analyze the degeneracy $d_{KK}(l'_0)$ associated with the low energy dynamics of the taub-NUT space, carrying momentum $-l'_0$ along S^1. Under the duality transformation described in section 2 the Taub-NUT space gets mapped to a fundamental string wound once along S^1 and carrying momentum $-l'_0$ along S^1 in heterotic string theory on $(T^4 \times S^1)/\,\mathbb{Z}_N \times \hat{S}^1$. Since this is a twisted sector state, the problem of computing $d_{KK}(l'_0)$ reduces to the problem of computing the degeneracy of twisted sectors states in this string theory. This has been done in appendix A and the result is

$$\sum_{l'_0} d_{KK}(l'_0)e^{2\pi i l'_0 \hat{\rho}} = 16\, e^{-2\pi i \hat{\rho}} \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n \hat{\rho}})^{-\frac{24}{N+1}} (1 - e^{2\pi i n N \hat{\rho}})^{-\frac{24}{N+1}} \right\} \tag{3.2}$$

The Taub-NUT space breaks eight of the sixteen supersymmetry generators in type IIB string theory on K3. This gives rise to eight fermonic zero modes. The factor of 16 in (3.2) arises from the quantization of these zero modes.

Next we turn to the degeneracy $d_{CM}(l_0, j_0)$ associated with the overall motion of the D1-D5-system in Taub-NUT space carrying momentum $-l_0$ along S^1 and j_0 along $\widetilde{S}^1$. The low energy theory describing the overall dynamics of the D1-D5 system in Taub-NUT space is that of a supersymmetric σ-model with Taub-NUT target space, the coordinate along S^1 being identified as the world-sheet σ coordinate of this supersymmetric field theory. Since $S^1/\,\mathbb{Z}_N$ has period $2\pi/N$, the natural unit of momentum along σ is N. As a result a BPS state of the D-brane system carrying momentum $-l_0$ correponds to $L_0 = l_0/N$, $\bar{L}_0 = 0$.[4] On the other hand the momentum j_0 along $\widetilde{S}^1$ is the U(1) charge associated with the angular direction of the Taub-NUT space which becomes the compact circle asymptotically. The degeneracy of such states has been computed in appendix B, and the result is

$$\sum_{l_0, j_0} d_{CM}(l_0, j_0)e^{2\pi i l_0 \hat{\rho} + 2\pi i j_0 \hat{v}} = 4\, e^{-2\pi i \hat{v}} (1 - e^{-2\pi i \hat{v}})^{-2}$$

$$\prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n N \hat{\rho}})^4 (1 - e^{2\pi i n N \hat{\rho} + 2\pi i \hat{v}})^{-2} (1 - e^{2\pi i n N \hat{\rho} - 2\pi i \hat{v}})^{-2} \right\}.$$

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[4]Throughout this paper L_0 and $\bar{L}_0$ of a state in a CFT will denote the left and right-moving world-sheet momenta with a normalization in which the world-sheet coordinate σ has period 2π. In RR sector we implicitly subtract the factors of $c/24$ ($\bar{c}/24$) from L_0 ($\bar{L}_0$) which arise in going from the sphere to the cylinder coordinates.

The D1-D5 system in K3×Taub-NUT space breaks four of the eight supersymmetry generators of type IIB string theory on $K3 \times$ Taub-NUT. Quantization of the associated zero modes gives rise to the factor of 4 in (3.3). The factor of 16 in (3.2) and 4 in (3.3) together provides the 64-fold degeneracy of a 1/4 BPS supermultiplet. As pointed out earlier, these 64 states contain equal numbers of bosons and fermions. After factoring this out we count the rest of the states with weight $+1$ for bosons and weight -1 for fermions.

Let us now turn to the computation of the degeneracy $n(w, l, j)$ of a single D1-brane moving inside the D5-brane, carrying winding w, momentum $-l$ and angular momentum j. We denote by σ the coordinate along the length of the D1-brane, σ being normalized so that it coincides with the target space coordinate in which the original S^1 had period 2π. Since the D1-brane carries winding charge w, σ changes by $2\pi w/N$ as we traverse the whole length of the string, regarded as a configuration in the orbifold. Under $\sigma \to \sigma + 2\pi w/N$, the physical coordinate of the D1-brane shifts by $2\pi r/N$ along S^1 where

$$r = w \mod N. \tag{3.4}$$

$\mathbb{Z}_N$ invariance then requires that under $\sigma \to \sigma + 2\pi w/N$ the location of the D1-brane along K3 gets transformed by $\widetilde{g}^r = \widetilde{g}^w$.

We expect the low energy dynamics of this D-brane system to be described by a superconformal field theory (SCFT) with target space $K3$ subject to the above boundary condition. Since the D1-brane is subject to twisted boundary condition with period $2\pi w/N$, the natural unit of momentum on the brane is N/w. Thus a total momentum $-l$ corresponds to $-lw/N$ unit of momentum and can be identified with the $\bar{L}_0 - L_0$ eigenvalue of the state. Since the BPS condition on the D1-brane corresponds to $\bar{L}_0 = 0$ we have $L_0 = lw/N$. Thus we are looking for a state in the SCFT with

$$L_0 = lw/N, \qquad \bar{L}_0 = 0. \tag{3.5}$$

The bosonic and fermionic excitations on the brane satisfy identical boundary conditions in order to preserve space-time supersymmetry. Hence the state belongs to the Ramond-Ramond (RR) sector. Furthermore the boundary condition described below eq. (3.4) tells us that the state is twisted by $\widetilde{g}^r$. Finally, since the total momentum along S^1 is $-l$, under translation by $2\pi/N$ along S^1 this state picks up a phase $e^{-2\pi i l/N}$. Thus the projection operator onto $\mathbb{Z}_N$ invariant states is given by

$$\frac{1}{N} \sum_{s=0}^{N-1} e^{-2\pi i s l/N} \widetilde{g}^s. \tag{3.6}$$

Putting these results together we see that the total number of $\mathbb{Z}_N$ invariant bosonic minus fermionic states of the single D1-brane carrying quantum numbers w, l, j is given by

$$n(w, l, j) \equiv \frac{1}{N} \sum_{s=0}^{N-1} e^{-2\pi i s l/N} Tr_{RR;\widetilde{g}^r} \left(\widetilde{g}^s (-1)^{F+\bar{F}} \delta_{NL_0, lw} \delta_{\mathcal{J}, j} \right). \tag{3.7}$$

where $Tr_{RR;\widetilde{g}^r}$ denotes trace over RR sector states twisted by $\widetilde{g}^r$, $F, \bar{F}$ are world-sheet fermion numbers (which coincide with the space-time fermion numbers) associated with the left and right-moving excitations on the D1-brane, and $\mathcal{J}$ is the U(1) generator associated with an appropriate R-symmetry current of this conformal field theory[5] which can be identified with the angular momentum operator [16]. Insertion of $(-1)^{\bar{F}}$ in the trace automatically projects onto $\bar{L}_0 = 0$ states.

Let us define [8]

$$F^{(r,s)}(\tau, z) \equiv \frac{1}{N} Tr_{RR;\widetilde{g}^r} \left(\widetilde{g}^s (-1)^{F+\bar{F}} e^{2\pi i \tau L_0} e^{2\pi i \mathcal{J} z} \right) \tag{3.8}$$

Then we have [8]

$$F^{(0,0)}(\tau, z) = \frac{8}{N} A(\tau, z) \,,$$

$$F^{(0,s)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) - \frac{2}{N+1} B(\tau, z) E_N(\tau) \qquad \text{for } 1 \le s \le (N-1) \,,$$

$$F^{(r,rk)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) + \frac{2}{N(N+1)} E_N \left(\frac{\tau + k}{N} \right) B(\tau, z) \,,$$
$$\text{for } 1 \le r \le (N-1), \ 0 \le k \le (N-1) \,, \tag{3.9}$$

where

$$A(\tau, z) = \left[\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right] \,, \tag{3.10}$$

$$B(\tau, z) = \eta(\tau)^{-6} \vartheta_1(\tau, z)^2 \,, \tag{3.11}$$

and

$$E_N(\tau) = \frac{12i}{\pi(N-1)} \partial_\tau \left[\ln \eta(\tau) - \ln \eta(N\tau) \right] = 1 + \frac{24}{N-1} \sum_{\substack{n_1, n_2 \ge 1 \\ n_1 \ne 0 \bmod N}} n_1 e^{2\pi i n_1 n_2 \tau} \,. \tag{3.12}$$

$F^{(r,s)}(\tau, z)$ has power series expansion of the form [8]

$$F^{(r,s)}(\tau, z) = \sum_{n \in \ \mathbb{Z}/N, j \in \ \mathbb{Z}} c^{(r,s)}(4n - j^2) e^{2\pi i n \tau} e^{2\pi i j z} \,, \tag{3.13}$$

for appropriate coefficients $c^{(r,s)}(4n - j^2)$. From (3.8), (3.13) it follows that

$$\frac{1}{N} tr_{RR,\widetilde{g}^r} \left(\widetilde{g}^s (-1)^{F+\bar{F}} \delta_{NL_0, lw} \delta_{\mathcal{J}, j} \right) = c^{(r,s)}(4lw/N - j^2) \,. \tag{3.14}$$

Hence (3.7) gives

$$n(w, l, j) = \sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(r,s)}(4lw/N - j^2) \,, \qquad r = w \bmod N \,. \tag{3.15}$$

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[5]In this case the full R-symmetry group is $\text{SO}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R$ and $\mathcal{J}$ corresponds to twice the $\text{U}(1)_L$ generator of the $\text{SU}(2)_L$ group.

Using the result for single D1-brane spectrum, we can now evaluate the degeneracy of multiple D1-branes moving inside the D5-brane. Let $d_{D1}(W, L, J')$ denote the degeneracy of this system, carrying total D1-brane charge $W = \sum_i w_i$, total momentum $-L = -\sum_i l_i$ along S^1 and total angular momentum $J' = \sum_i j_i$. Then a straightforward combinatoric analysis shows that

$$\sum_{W,L,J'} d_{D1}(W, L, J') e^{2\pi i(\hat\sigma W/N + \hat\rho L + \hat v J')} = \prod_{\substack{w,l,j \in \mathbb{Z} \\ w>0, l\geq 0}} \left(1 - e^{2\pi i(\hat\sigma w/N + \hat\rho l + \hat v j)}\right)^{-n(w,l,j)} . \quad (3.16)$$

Let us now turn to the full system containing the D5-brane and multiple D1-branes in the Kaluza-Klein monopole background. The combinatoric problem to be solved is the following. We have total quantum numbers (Q_1, n, J), to be distributed among multiple D1-branes moving inside a D5-brane, the overall D1-D5 system and the Taub-NUT space according to the relation (3.1). The degeneracy associated with the dynamics of (multiple) D1-branes inside the D5-brane, carrying quantum numbers $(W = \sum w_i, L = \sum l_i, J' = \sum j_i)$ is given by $d_{D1}(W, L, J')$, the degeneracy associated with the overall dynamics of the D1-D5 system carrying quantum numbers l_0, j_0 is given by $d_{CM}(l_0, j_0)$ and the degeneracy associated with the dynamics of the Taub-NUT space carrying quantum number l_0' is given by $d_{CM}(l_0')$. If $h(Q_1, n, J)$ denotes the total number of bosonic minus fermionic supermultiplets (in the sense described earlier) of the mutiple D1-brane system carrying quantum numbers (Q_1, n, J) and if we define

$$f(\hat\rho, \hat\sigma, \hat v) = \sum_{Q_1, n, J} h(Q_1, n, J) e^{2\pi i(\hat\rho n + \hat\sigma(Q_1 - 1)/N + \hat v J)} , \quad (3.17)$$

then we get

$$f(\hat\rho, \hat\sigma, \hat v) = \frac{1}{64} e^{-2\pi i\hat\sigma/N} \sum_{W,L,J'} d_{D1}(W, L, J') e^{2\pi i(\hat\sigma W/N + \hat\rho L + \hat v J')}$$
$$\left(\sum_{l_0, j_0} d_{CM}(l_0, j_0) e^{2\pi i l_0 \hat\rho + 2\pi i j_0 \hat v}\right) \left(\sum_{l_0'} d_{KK}(l_0') e^{2\pi i l_0' \hat\rho}\right) .$$
$$(3.18)$$

The factor of $1/64$ in this equation arises due to the fact that a single 1/4 BPS supermultiplet is 64-fold degenerate; thus in order to count the number of supermultiplets we need to divide the total number of states by 64. Using (3.2), (3.3), (3.15), (3.16) and the relations [8]

$$c^{(0,0)}(0) = \frac{20}{N} , \qquad c^{(0,0)}(-1) = \frac{2}{N} ,$$
$$c^{(0,s)}(0) = \frac{1}{N}\left(20 - \frac{24N}{N+1}\right) , \qquad c^{(0,s)}(-1) = \frac{2}{N} , \quad \text{for } s = 1, 2, \ldots (N-1) ,$$
$$(3.19)$$

one can reduce (3.18) to

$$f(\hat{\rho}, \hat{\sigma}, \hat{v}) = \tag{3.20}$$

$$= e^{-2\pi i(\hat{\rho}+\hat{\sigma}/N+\hat{v})} \prod_{r=0}^{N-1} \prod_{\substack{k' \in \mathbb{Z}+\frac{r}{N}, l, j \in \mathbb{Z} \\ k', l \geq 0, j < 0 \text{ for } k'=l=0}} \left(1 - e^{2\pi i(\hat{\sigma}k'+\hat{\rho}l+\hat{v}j)}\right)^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(r,s)}(4lk'-j^2)} .$$

The $k' = 0$ term in the last expression comes from the terms involving $d_{CM}(l_0, j_0)$ and $d_{KK}(l'_0)$. Comparing the right hand side of this equation with the expression for $\widetilde{\Phi}_k$ given in (C.2) we can rewrite (3.18) as

$$f(\hat{\rho}, \hat{\sigma}, \hat{v}) = -\frac{(i\sqrt{N})^{-k-2}}{\widetilde{\Phi}_k(\hat{\rho}, \hat{\sigma}, \hat{v})} . \tag{3.21}$$

$\widetilde{\Phi}_k$ is a weight k modular form of an appropriate subgroup of the Siegel modular group [8].

From (3.17), (3.21) it follows that for $n > 0$, $Q_1 > 1$, $J > 0$,

$$h(Q_1, n, J) = K \int_{\hat{C}} d\hat{\rho} d\hat{\sigma} d\hat{v} \frac{1}{\widetilde{\Phi}_k(\hat{\rho}, \hat{\sigma}, \hat{v})} \exp\left[-2\pi i(\hat{\rho}n + \hat{\sigma}(Q_1 - 1)/N + \hat{v}J)\right] , \tag{3.22}$$

where

$$K = -\frac{1}{N}(i\sqrt{N})^{-k-2} , \tag{3.23}$$

and the integration surface $\hat{C}$ is the three real dimensional surface:

$$Im \, \hat{\rho} = M_2/N, \quad Im \, \hat{\sigma} = M_1 N, \quad Im \, \hat{v} = M_3,$$
$$0 \leq Re \, \hat{\rho} \leq 1, \quad 0 \leq Re \, \hat{\sigma} \leq N, \quad 0 \leq Re \, \hat{v} \leq 1, \tag{3.24}$$

M_1, M_2 and M_3 being three large but fixed positive numbers.

Now recall that from the map described in section 2, we have for this system

$$Q_e^2 = 2n/N, \qquad Q_m^2 = 2(Q_1 - Q_5)Q_5 = 2(Q_1 - 1), \qquad J = Q_e \cdot Q_m . \tag{3.25}$$

Thus the degeneracy $d(Q_e, Q_m)$ of the four dimensional black hole is related to the quantity $h(Q_1, n, J)$ given via eqs. (3.17), (3.21) as

$$d(Q_e, Q_m) = h\left(\frac{1}{2}Q_m^2 + 1, \frac{1}{2}Q_e^2 N, Q_e \cdot Q_m\right)$$
$$= K \int_{\widetilde{C}} d\hat{\rho} d\hat{\sigma} d\hat{v} \frac{1}{\widetilde{\Phi}_k(\hat{\rho}, \hat{\sigma}, \hat{v})} \exp\left[-i\pi(\hat{\rho}Q_e^2 N + \hat{\sigma}Q_m^2/N + 2\hat{v}Q_e \cdot Q_m)\right] . \tag{3.26}$$

It has been shown in appendix C that

$$\widetilde{\Phi}_k(\hat{\rho}, \hat{\sigma}, \hat{v}) = \widetilde{\Phi}_k(\hat{\sigma}/N, \hat{\rho}N, \hat{v}) . \tag{3.27}$$

Thus defining new variables:

$$\widetilde{\rho} = \hat{\sigma}/N, \quad \widetilde{\sigma} = \hat{\rho}N, \quad \widetilde{v} = \hat{v}, \tag{3.28}$$

we can write

$$d(Q_e, Q_m) = K \int_C d\widetilde{\rho}d\widetilde{\sigma}d\widetilde{v} \frac{1}{\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})} \exp\left[-i\pi(\widetilde{\rho}Q_m^2 + \widetilde{\sigma}Q_e^2 + 2\widetilde{v}Q_e \cdot Q_m)\right], \tag{3.29}$$

with the integration surface C being the three real dimensional surface:

$$Im\,\widetilde{\rho} = M_1, \quad Im\,\widetilde{\sigma} = M_2, \quad Im\,\widetilde{v} = M_3,$$
$$0 \leq Re\,\widetilde{\rho} \leq 1, \quad 0 \leq Re\,\widetilde{\sigma} \leq N, \quad 0 \leq Re\,\widetilde{v} \leq 1. \tag{3.30}$$

This agrees with the proposal put forward in [7−9].

Although this formula has been derived for $Q_5 = 1$, a more systematic analysis based on mapping the low energy dynamics of the D1-D5 system to a conformal field theory with the symmetric product of $(Q_1 - Q_5)Q_5 + 1$ copies of $K3$ as the target space [24] can be used to show that the final formula holds for general Q_5. This has been done in appendix D.

4. Asymptotic behaviour of the degeneracy of dyons

In this section we shall develop a systematic method for computing the behaviour of $d(Q_e, Q_m)$ given in (3.29) for large charges. As in [1, 2, 7, 25] we shall try to estimate the behaviour of this integral for large charges by deforming the integration contour and picking up residues from the poles. This removes one of the three integrals. The remaining two integrals will then be performed using a saddle point approximation. At the end of this process we need to compare the contributions from different poles and identify the one that gives the dominant contribution in the large charge limit.

The poles of the integrand in (3.29) arise from the zeroes of $\widetilde{\Phi}_k$. The locations of the zeroes and poles of $\widetilde{\Phi}_k$ have been determined in appendix E. According to eqs. (E.16), (E.19), $\widetilde{\Phi}_k$ has second order zeroes at

$$n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 = 0,$$
$$\text{for} \quad m_1 \in N\,\mathbb{Z}, n_1 \in \mathbb{Z}, b \in 2\,\mathbb{Z} + 1, m_2, n_2 \in \mathbb{Z}, \quad m_1n_1 + m_2n_2 + \frac{b^2}{4} = \frac{1}{4}. \tag{4.1}$$

Let us define

$$A = n_2, \quad B = (n_1, -m_1, \frac{1}{2}b), \quad y = (\widetilde{\rho}, \widetilde{\sigma}, -\widetilde{v}), \qquad C = m_2, \quad q = (Q_e^2, Q_m^2, Q_e \cdot Q_m), \tag{4.2}$$

and denote by $\cdot$ the SO(2, 1) invariant inner product

$$(x^1, x^2, x^3) \cdot (y^1, y^2, y^3) = x^1y^2 + x^2y^1 - 2x^3y^3. \tag{4.3}$$

Then we have

$$y^2 \equiv y \cdot y = 2(\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2), \qquad B \cdot y = b\widetilde{v} + n_1\widetilde{\sigma} - m_1\widetilde{\rho}, \tag{4.4}$$

and the first equation of (4.1) may be rewritten as

$$\frac{1}{2}Ay^2 + B \cdot y + C = 0. \tag{4.5}$$

Picking up residue at the pole forces us to evaluate the exponent in (3.29)

$$- i\pi \left(\widetilde{\rho}Q_m^2 + \widetilde{\sigma}Q_e^2 + 2\widetilde{v}Q_e \cdot Q_m \right) = -i\,\pi\, q \cdot y, \tag{4.6}$$

at (4.5). To leading approximation the location of the saddle point is now determined by extremizing (4.6) with respect to y subject to the condition (4.5). This gives

$$q + \lambda(Ay + B) = 0, \tag{4.7}$$

where λ is a lagrange multiplier. (4.5) and (4.7) now give:

$$\lambda = \pm\sqrt{\frac{q^2}{B^2 - 2AC}}, \qquad y = -\frac{1}{A}\left(\frac{q}{\lambda} + B\right). \tag{4.8}$$

Since

$$B^2 - 2AC = -2(m_1 n_1 + m_2 n_2 + \frac{b^2}{4}) = -\frac{1}{2} \tag{4.9}$$

due to the last equation in (4.1), we get

$$\lambda = \pm\sqrt{-2q^2}, \tag{4.10}$$

Thus at the saddle point the exponential $e^{-i\pi q \cdot y}$ takes the form:

$$E \equiv e^{-i\pi\, q \cdot y} = e^{i\pi(q^2/A\lambda + q \cdot B/A)} = e^{\left(\pm\pi\sqrt{q^2/2} + i\pi q \cdot B\right)/A}. \tag{4.11}$$

Now since $q \cdot B$ and A are integers, the second term only gives a phase. Hence

$$|E| = e^{\frac{\pi}{A}\sqrt{q^2/2}} = e^{\pi\sqrt{Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2}/n_2}. \tag{4.12}$$

Note that we have chosen the sign of the square root so that the sign in the exponent is positive since this gives the dominant contribution.

Eq. (4.12) shows that the leading contribution to the integral comes from the saddle point corresponding to $n_2 = 1$. In this case a $\widetilde{\rho} \to \widetilde{\rho} + 1$ transformation in (4.1) induces $n_1 \to n_1 + 1$, $m_2 \to m_2 - m_1$, and since $n_1 \in \mathbb{Z}$, we can use this symmetry to bring the saddle point to $n_1 = 0$. On the other hand a $\widetilde{\sigma} \to \widetilde{\sigma} + N$ transformation in eq. (4.1) induces $m_1 \to m_1 - N$, $m_2 \to m_2 + n_1 N$. Since $m_1 \in N\,\mathbb{Z}$, we can use this transformation to bring m_1 to 0. Finally the $\widetilde{v} \to \widetilde{v} + 1$ transformation in (4.1) induces $b \to b - 2$, $m_2 \to m_2 + b - 1$. Since $b \in 2\,\mathbb{Z} + 1$, we can use this transformation to set $b = 1$. m_2 is now determined to be zero from the last equation in (4.1). Thus we have

$$m_1 = m_2 = n_1 = 0, \quad n_2 = 1, \quad b = 1. \tag{4.13}$$

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The corresponding zero of $\widetilde{\Phi}_k$ is at

$$\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2 + \widetilde{v} = 0\,. \tag{4.14}$$

The exponent of eq. (4.12) for $n_2 = 1$ gives the leading term in the expression for the statistical entropy, but in order to find a systematic expansion of the entropy in inverse power of charges, we need to carefully evaluate the complete contribution from the pole at (4.14). For this it will be useful to define new variable ρ, σ, v through the relations:

$$\rho = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2}{\widetilde{\sigma}}, \qquad \sigma = \frac{\widetilde{\rho}\widetilde{\sigma} - (\widetilde{v} - 1)^2}{\widetilde{\sigma}}, \qquad v = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v}}{\widetilde{\sigma}}\,, \tag{4.15}$$

or equivalently,

$$\widetilde{\rho} = \frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \qquad \widetilde{\sigma} = \frac{1}{2v - \rho - \sigma}, \qquad \widetilde{v} = \frac{v - \rho}{2v - \rho - \sigma}\,. \tag{4.16}$$

In these variables (4.14) gets mapped to $v = 0$, and we have, near $v = 0$ [7],

$$\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = 4\pi^2 \left(2v - \rho - \sigma\right)^k v^2 f^{(k)}(\rho)f^{(k)}(\sigma) + \mathcal{O}(v^4)\,,$$
$$f^{(k)}(\tau) \equiv \eta(\tau)^{k+2}\eta(N\tau)^{k+2}\,. \tag{4.17}$$

Also it follows from (4.15), (4.16) that

$$d\widetilde{\rho}\,d\widetilde{\sigma}\,d\widetilde{v} = (2v - \rho - \sigma)^{-3}\,d\rho\,d\sigma\,dv\,. \tag{4.18}$$

Thus the contribution to (3.29) from the pole at $\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0$ is given by

$$d(Q_e, Q_m) \simeq \frac{K}{4\pi^2} \int_{C'} d\rho\,d\sigma\,dv\, v^{-2}\left(2v - \rho - \sigma\right)^{-k-3}\left(f^{(k)}(\rho)f^{(k)}(\sigma)\right)^{-1}$$
$$\exp\left[-i\pi\left\{\frac{v^2 - \rho\sigma}{2v - \rho - \sigma}Q_m^2 + \frac{1}{2v - \rho - \sigma}Q_e^2 + \frac{2(v - \rho)}{2v - \rho - \sigma}Q_e \cdot Q_m\right\}\right]\,, \tag{4.19}$$

where the integration contour C' now encloses the divisor $v = 0$. The correction to this formula involves contribution from other poles for which $n_2 \neq 1$, and are suppressed by powers of e^{-Q^2}. Evaluating the v integral in (4.19) by Cauchy's formula, we get

$$d(Q_e, Q_m) \simeq i\frac{K}{2\pi}\left(-1\right)^{k+Q_e\cdot Q_m}\int \frac{d\rho\,d\sigma}{(\rho + \sigma)^2}$$
$$\left[-2(k + 3) + 2\pi i\left\{\frac{\rho\sigma}{\rho + \sigma}Q_m^2 - \frac{1}{\rho + \sigma}Q_e^2 + \frac{\rho - \sigma}{\rho + \sigma}Q_e \cdot Q_m\right\}\right]$$
$$\exp\left[-i\pi\left\{\frac{\rho\sigma}{\rho + \sigma}Q_m^2 - \frac{1}{\rho + \sigma}Q_e^2 + \frac{\rho - \sigma}{\rho + \sigma}Q_e \cdot Q_m\right\}\right.$$
$$\left. -\ln f^{(k)}(\rho) - \ln f^{(k)}(\sigma) - (k + 2)\ln(\rho + \sigma)\right]\,. \tag{4.20}$$

Let us now introduce new complex variables a and S through the relations:

$$\rho = a + iS, \qquad \sigma = -a + iS. \tag{4.21}$$

Then (4.20) may be rewritten as

$$
\begin{aligned}
d(Q_e, Q_m) \simeq -\frac{K}{4\pi} (-1)^{k+Q_e \cdot Q_m} \int \frac{dS\, da}{S^2} \\
\left[2(k+3) + \frac{\pi}{S} \left\{ (a^2 + S^2)Q_m^2 + Q_e^2 - 2a Q_e \cdot Q_m \right\} \right] \\
\exp \left[\frac{\pi}{2S} \left\{ (a^2 + S^2)Q_m^2 + Q_e^2 - 2a Q_e \cdot Q_m \right\} \right. \\
\left. - \ln f^{(k)}(a + iS) - \ln f^{(k)}(-a + iS) - (k+2)\ln(2iS) \right].
\end{aligned}
\tag{4.22}
$$

So far we have not specified the contour for a and S integration, except that it must pass through the saddle point. In the leading approximation the saddle point, obtained by extremizing

$$\frac{\pi}{2S} \left\{ (a^2 + S^2)Q_m^2 + Q_e^2 - 2a Q_e \cdot Q_m \right\} \tag{4.23}$$

occurs for real values of a and S. Thus we can take the contour of integration to be along the real a and S axis. If we now define

$$\tau = -a + iS \equiv \tau_1 + i\tau_2, \tag{4.24}$$

then (4.22) may be reexpressed as

$$
\begin{aligned}
d(Q_e, Q_m) \simeq K_0 (-1)^{Q_e \cdot Q_m} \int \frac{d^2\tau}{\tau_2^2} \left[2(k+3) + \frac{\pi}{\tau_2} |Q_e + \tau Q_m|^2 \right] \\
\exp \left[\frac{\pi}{2\tau_2} |Q_e + \tau Q_m|^2 - \ln f^{(k)}(\tau) - \ln f^{(k)}(-\bar\tau) - (k+2)\ln(2\tau_2) \right],
\end{aligned}
\tag{4.25}
$$

where

$$K_0 = -\frac{K}{4\pi}(i)^{-k-2}(-1)^k = \frac{1}{4\pi\, N^{(k+4)/2}}. \tag{4.26}$$

Note that the degeneracy factor is positive or negative depending on whether $Q_e \cdot Q_m$ is positive or negative. This is natural from the point of view of a black hole solution since a classical black hole is expected to be bosonic (fermionic) for $Q_e \cdot Q_m$ even (odd).

Identifying $|d(Q_e, Q_m)|$ with $e^{S_{stat}(Q_e, Q_m)}$ where S_{stat} denotes the statistical entropy, we can rewrite (4.25) as

$$e^{S_{stat}(Q_e, Q_m)} = \int \frac{d^2\tau}{\tau_2^2} e^{-F(\vec\tau)}, \tag{4.27}$$

where $\vec\tau = (\tau_1, \tau_2)$ or $(\tau, \bar\tau)$ depending on the basis we choose to use, and

$$
\begin{aligned}
F(\vec\tau) = - \left[\frac{\pi}{2\tau_2} |Q_e + \tau Q_m|^2 - \ln f^{(k)}(\tau) - \ln f^{(k)}(-\bar\tau) - (k+2)\ln(2\tau_2) \right. \\
\left. + \ln \left\{ K_0 \left(2(k+3) + \frac{\pi}{\tau_2} |Q_e + \tau Q_m|^2 \right) \right\} \right],
\end{aligned}
\tag{4.28}
$$

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Note that $F(\vec{\tau})$ also depends on the charge vectors Q_e, Q_m, but we have not explicitly displayed these in its argument. In (4.27) the integration measure $d^2\tau/(\tau_2)^2$ as well as the integrand $e^{F(\vec{\tau})}$ are manifestly invariant under the $\Gamma_1(N)$ transformation:

$$Q_e \to aQ_e - bQ_m, \quad Q_m \to -cQ_e + dQ_m, \quad \tau \to \frac{a\tau + b}{c\tau + d},$$
$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad a, d = 1 \bmod N, \quad c = 0 \bmod N. \tag{4.29}$$

Thus $S_{stat}(Q_e, Q_m)$ computed from (4.27) is invariant under $\Gamma_1(N)$.

We shall now describe a systematic procedure for evaluating S_{stat} as an expansion in inverse powers of the charges. For this we introduce the generating function:

$$e^{W(\vec{J})} = \int \frac{d^2\tau}{\tau_2^2} e^{-F(\vec{\tau}) + \vec{J} \cdot \vec{\tau}}, \tag{4.30}$$

for a two dmensional vector $\vec{J}$, and define $\Gamma(\vec{u})$ as the Legendre transform of $W(\vec{J})$:

$$\Gamma(\vec{u}) = \vec{J} \cdot \vec{u} - W(\vec{J}), \qquad u_i = \frac{\partial W(\vec{J})}{\partial J_i}. \tag{4.31}$$

It follows from (4.31) that

$$J_i = \frac{\partial \Gamma(\vec{u})}{\partial u_i}. \tag{4.32}$$

As a result if

$$\frac{\partial \Gamma(\vec{u})}{\partial \vec{u}_i} = 0 \quad \text{at } \vec{u} = \vec{u}_0, \tag{4.33}$$

then it follows from (4.30)- (4.32), (4.27) that

$$\Gamma(\vec{u}_0) = -W(\vec{J} = 0) = -S_{stat}. \tag{4.34}$$

Thus the computation of S_{stat} can be done by first calculating $\Gamma(\vec{u})$ and then evaluating it at its extremum. $\Gamma(\vec{u})$ in turn can be calculated by regarding this as a sum of one particle irreducible (1PI) Feynman diagrams in the zero dimensional field theory with action $F(\vec{\tau}) + 2\ln\tau_2$. Since S_{stat} is given by the value of the function $-\Gamma(\vec{u})$ at its extremum, we can identify $-\Gamma(\vec{u})$ as the entropy function for the statistical entropy in analogy with the corresponding result for black hole entropy [26, 27].

A convenient method of calculating $\Gamma(\vec{u})$ is the so called background field method. For this we choose some arbitrary base point $\vec{\tau}_B$ and define

$$e^{W_B(\vec{\tau}_B, \vec{J})} = \int \frac{d^2\eta}{(\tau_{B2} + \eta_2)^2} e^{-F(\vec{\tau}_B + \vec{\eta}) + \vec{J} \cdot \vec{\eta}}, \tag{4.35}$$

$$\Gamma_B(\vec{\tau}_B, \vec{\chi}) = \vec{J} \cdot \vec{\chi} - W_B(\vec{\tau}_B, \vec{J}), \qquad \chi_i = \frac{\partial W_B(\vec{\tau}_B, \vec{J})}{\partial J_i}. \tag{4.36}$$

By shifting the integration variable in (4.35) to $\vec{\tau} = \vec{\tau}_B + \vec{\eta}$ it follows easily that

$$W_B(\vec{\tau}_B, \vec{J}) = W(\vec{J}) - \vec{\tau}_B \cdot \vec{J}, \tag{4.37}$$

and hence

$$\Gamma_B(\vec{\tau}_B, \vec{\chi}) = \Gamma(\vec{\tau}_B + \vec{\chi}). \tag{4.38}$$

Thus the computation of $\Gamma(\vec{u})$ reduces to the computation of $\Gamma_B(\vec{u}, \vec{\chi} = 0)$. The latter in turn can be computed as the sum of 1PI vacuum diagrams in the 0-dimensional field theory with action $F(\vec{u} + \vec{\eta}) + 2\ln(u_2 + \eta_2)$, with $\vec{\eta}$ regarded as fundamental fields, and $\vec{u}$ regarded as some fixed background.

While this gives a definition of the statistical entropy function whose extremization leads to the statistical entropy, the entropy function constructed this way is not manifestly duality invariant. This is due to the fact that since the duality transformation has a non-linear action on (τ_1, τ_2), the generating function $W(\vec{J})$ defined in (4.30) and hence also the effective action $\Gamma(\vec{u})$ defined in (4.31) does not have manifest duality symmetries. Of course the statistical entropy obtained by extremizing $\Gamma(\vec{u})$ will be duality invariant since this is given in terms of the manifestly duality invariant integral (4.28). In appendix F we have described a slightly different construction based on Riemann normal coordinates which yields a manifestly duality invariant statistical entropy function. The result of this analysis is that instead of using the function $-\Gamma_B(\vec{\tau})$ as the statistical entropy function we can use a different manifestly duality invariant function $-\widetilde{\Gamma}_B(\vec{\tau})$ as the statistical entropy function. $\widetilde{\Gamma}_B(\vec{\tau}_B)$ is defined as the sum of 1PI vacuum diagrams computed from the action

$$-\ln\left(\frac{1}{|\vec{\xi}|}\sinh|\vec{\xi}|\right) - \sum_{n=0}^{\infty}\frac{1}{n!}(\tau_{B2})^n \xi_{i_1}\ldots\xi_{i_n}\, D_{i_1}\cdots D_{i_n}F(\vec{\tau})\bigg|_{\vec{\tau}=\vec{\tau}_B}, \tag{4.39}$$

where D_τ, $D_{\bar{\tau}}$ are duality invariant covariant derivatives defined recursively through the relation:

$$D_\tau(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) = (\partial_\tau - im/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})),$$
$$D_{\bar{\tau}}(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) = (\partial_{\bar{\tau}} + in/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})), \tag{4.40}$$

for any arbitrary ordering of D_τ and $D_{\bar{\tau}}$ in $D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})$. During this computation the components $(\xi, \bar{\xi})$ or (ξ_1, ξ_2) of $\vec{\xi}$ are to be regarded as the zero dimensional quantum fields and $\vec{\tau}_B$ is to be taken as a fixed base point. The result of this computation expresses $\widetilde{\Gamma}_B(\vec{\tau}_B)$ in terms of manifestly duality invariant quantity $F(\vec{\tau})$ and its duality invariant covariant derivatives.

It has also been shown in appendix F that explicit evaluation of $\widetilde{\Gamma}_B(\vec{\tau})$ gives

$$-\widetilde{\Gamma}_B(\vec{\tau}) = \frac{\pi}{2\tau_2}|Q_e + \tau Q_m|^2 - \ln f^{(k)}(\tau) - \ln f^{(k)}(-\bar{\tau}) - (k+2)\ln(2\tau_2) + \text{constant} + \mathcal{O}(Q^{-2}). \tag{4.41}$$

Up to an additive constant this agrees with the black hole entropy function for CHL models calculated in [27–29] using the quantum effective action of these theories [30, 31].

A. Counting twisted sector elementary heterotic string states in CHL models

In this appendix we shall compute

$$g(\tau) = \sum_{l_0'} d_{KK}(l_0') e^{2\pi i l_0' \tau} \tag{A.1}$$

where $d_{KK}(l_0')$ is the degeneracy of the BPS states of a Kaluza-Klein monopole associated with the compact direction $\widetilde{S}^1$ carrying momentum $-l_0'$ along S^1. As discussed in section 3, under a duality transformation $d_{KK}(l_0')$ gets mapped to the degeneracy of BPS states of a fundamental string wound once along S^1 and carrying momentum $-l_0'$ along S^1 in heterotic string theory on $(T^4 \times S^1)/\mathbb{Z}_N \times \hat{S}^1$. Since we have a singly wound string along S^1, after orbifolding it becomes a twisted sector state. For computing $g(\tau)$ we can now proceed as in [32]. In heterotic string theory on T^6 the internal momenta belong to the Narain lattice embedded in a 28-dimensional Lorentzian space of signature (6,22) [33, 34]. Let us denote by g_H the generator of the $\mathbb{Z}_N$ transformation and by $V_\parallel$ the subspace of the 28-dimensional momentum space which transform non-trivially under g_H. The dimension of $V_\parallel$ is equal to the number of U(1) gauge fields which are projected out by the orbifolding procedure. This in turn is given by [7]

$$20 - 2k, \quad k = \frac{24}{N+1} - 2. \tag{A.2}$$

Let L_0' denote the contribution to the L_0 eigenvalue of a state from all the left-handed bosonic oscillators and also from the components of the momentum along $V_\parallel$, and $\bar{L}_0'$ denote the contribution to the $\bar{L}_0$ eigenvalue from the right-handed bosonic and fermionic oscillators.[6] Then we have

$$L_0 - \bar{L}_0 = L_0' - \bar{L}_0' - l_0'/N, \tag{A.3}$$

since $-l_0'/N$ is the contribution to $L_0 - \bar{L}_0$ from components of the internal momentum invariant under g_H. Now level matching condition tells us that $L_0 - \bar{L}_0 = 0$, whereas BPS condition requires that we have $\bar{L}_0' = 0$. Thus on these states

$$L_0' = l_0'/N. \tag{A.4}$$

This allows us to reexpress $g(\tau)$ as

$$g(\tau) = \frac{1}{N} \sum_{s=0}^{N-1} Tr'_{g_H} \left(g_H^s e^{2\pi i \tau N L_0'} \right), \tag{A.5}$$

where Tr'_{g_H} denotes trace over states twisted by g_H carrying all possible momenta along $V_\parallel$ and involving arbitrary excitation of left-moving bosonic oscillators. We can simplify this expression by noting that in the twisted sector of the orbifold theory the $\mathbb{Z}_N$ projection

---

[6] All the components of the right-handed momentum are invariant under g_H and hence they do not contribute to L_0'.

implements level matching. Since we have already implemented the level matching condition on the states via eq. (A.4), g_H acts trivially on these states. In particular the part of g_H that corresponds to $2\pi/N$ translation along S^1 gives a phase of $e^{-2\pi i l_0'/N} = e^{-2\pi i L_0'}$, and this cancels the phase coming from the part of g_H that acts on the twisted bosonic oscillators. Thus we can rewrite (A.5) as

$$g(\tau/N) = Tr'_{g_H} \left(e^{2\pi i \tau L_0'} \right) . \tag{A.6}$$

Since the L_0' in the exponent comes from oscillator contribution and components of momentum along $V_\parallel$, only the part of g_H twisting which involves the 16 left-moving coordinates and the coordinates of T^4 are relevant for this computation.

Some useful properties of $g(\tau/N)$ are listed below:

1. From (A.6) we see that $g(\tau/N)$ can be identified as an appropriate partition function with spin structure $(0,1)$ where the spin structure (s,r) represents twisting by g_H^r and an insertion of g_H^s in the trace. Since under a modular transformation $\tau \rightarrow (a\tau + b)/(c\tau + d)$ the spin structure (s,r) gets transformed to

$$\begin{pmatrix} s \\ r \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ r \end{pmatrix} , \tag{A.7}$$

we see that the subgroup of SL(2, $\mathbb{Z}$) that preserves the set of spin structures $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ mod N is determined by the requirement

$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad b = 0 \bmod N, \quad d = 1 \bmod N . \tag{A.8}$$

This describes the group $\Gamma^1(N)$. Thus we expect that $g(\tau/N)$ will transform as a modular form of an appropriate weight under the group $\Gamma^1(N)$.

2. The weight of the modular form may be determined as follows. First of all note that $g(\tau/N)$ receives contribution from the 24 left-moving bosonic oscillators. This contributes -12 to the weight of the modular form. On the other hand $g(\tau/N)$ also involves a sum over a $(20 - 2k)$ dimensional momentum lattice embedded in $V_\parallel$. This gives a contribution of $(10 - k)$ to the modular weight. Thus the net modular weight of $g(\tau/N)$ is

$$10 - k - 12 = -k - 2 = -\frac{24}{N+1} . \tag{A.9}$$

3. Since $g(\tau/N)$ receives contribution from only bosonic oscillators, each term in the expansion of $g(\tau/N)$ as a power series in $e^{2\pi i \tau/N}$ has positive coefficients and hence $g(\tau/N)$ does not vanish in the interior of the upper half plane.

4. For large Im(τ), $g(\tau/N)$ behaves as

$$g(\tau/N) \simeq 16 e^{-2\pi i \tau/N} , \tag{A.10}$$

where the exponent $-2\pi i \tau/N$ comes from the L_0' eigenvalue of $-1/N$ associated with the ground state of the (twisted) bosonic oscillators [7], and the factor of 16

counts the 8 Ramond and 8 Neveu-Schwarz sector states associated with the broken supersymmetry generators.

5. For small τ the behaviour of $g(\tau/N)$ is controlled by the large $Im(\tau)$ behaviour of the partition function with spin structure (1,0), i.e. by the trace of $g_H e^{2\pi i L'_0/\tau}$ over the untwisted sector states. Since the untwisted sector ground state has L'_0 eigenvalue -1, this gives:

$$g(\tau/N) \sim e^{2\pi i/\tau} \qquad \text{for } \tau \to 0 \,. \tag{A.11}$$

From this it follows that $g(\tau/N)^{-1}$ is a modular form of $\Gamma^1(N)$ of weight $(k+2)$ without any pole in the interior of the upper half plane, and behaves as

$$(g(\tau/N))^{-1} \simeq \frac{1}{16} e^{2\pi i \tau/N}\,, \quad \text{for large Im } \tau,$$
$$\sim e^{-2\pi i/\tau} \quad \text{for small } \tau\,. \tag{A.12}$$

Since $g(\tau/N)^{-1}$ vanishes at the cusps $\tau = n + i\infty$, $n \in \mathbb{Z}$, and $\tau = 0$, we conclude that it is a cusp form of $\Gamma^1(N)$ of weight $(k+2)$. This leads to a unique choice for $g(\tau/N)$ [35]:

$$g(\tau/N) = \frac{16}{f^{(k)}(\tau/N)} = 16\eta(\tau/N)^{-k-2}\eta(\tau)^{-k-2}\,. \tag{A.13}$$

This gives

$$g(\tau) = 16\, e^{-2\pi i \tau} \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n\tau})^{-\frac{24}{N+1}}(1 - e^{2\pi i nN\tau})^{-\frac{24}{N+1}} \right\} \,. \tag{A.14}$$

For $N = 1$ this reproduces the standard result $g(\tau) = 16\eta(\tau)^{-24}$ for the partition function of half BPS states in toroidally compactified heterotic string theory [36]. For $N = 2$ this result agrees with that of [37] obtained by explicit computation.

B. Counting states in the supersymmetric field theory with Taub-NUT target space

In this appendix we will analyze the overall motion of the D1-D5 system in the Taub-NUT (Kaluza-Klein monopole) space and count the number of states of this system which carry momenta $-l_0$ along S^1 and j_0 along $\widetilde{S}^1$. The D1-D5-system wrapped on $K3 \times S^1$ in flat transverse space has four bosonic zero modes labelling the transverse coordinates and eight fermionic zero modes associated with the breaking of eight out of sixteen supersymmetries of type IIA string theory on $K3 \times S^1$. Thus when the transverse space is Taub-NUT, we expect the low energy dynamics of this system to be described by a $(1+1)$ dimensional supersymmetric field theory with four bosonic and eight fermionic coordinates, with the bosonic coordinates taking value in the Taub-NUT target space. Since the world-sheet coordinate σ of this field theory is identified with the coordinate along S^1, and since $S^1/\mathbb{Z}_N$ has periodicity $2\pi/N$, the natural unit of momentum along σ is N. As a result a state of the D-brane system carrying momentum $-l_0$ corresponds to a state in this field theory with $L_0 - \bar{L}_0 = l_0/N$.

We begin our analysis of this field theory by writing down the metric of the Taub-NUT space:

$$ds^2 = \left(1 + \frac{R}{r}\right)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + R^2\left(1 + \frac{R}{r}\right)^{-1}(2\,d\psi + \cos\theta d\phi)^2 \quad \text{(B.1)}$$

with the identifications:

$$(\theta, \phi, \psi) \equiv (2\pi - \theta, \phi + \pi, \psi + \frac{\pi}{2}) \equiv (\theta, \phi + 2\pi, \psi + \pi) \equiv (\theta, \phi, \psi + 2\pi). \quad \text{(B.2)}$$

Close to the origin the metric reduces to that of flat space R^4 written in terms of Euler angles θ, ϕ, ψ and r, while for large r it is that of $R^3 \times \widetilde{S}^1$, with $\widetilde{S}^1$ parametrized by the angular coordinate ψ. For later use it will be useful to introduce the cartesian coordinates:

$$x^1 = 2\sqrt{r}\cos\frac{\theta}{2}\cos\left(\psi + \frac{\phi}{2}\right), \qquad x^2 = 2\sqrt{r}\cos\frac{\theta}{2}\sin\left(\psi + \frac{\phi}{2}\right),$$

$$x^3 = 2\sqrt{r}\sin\frac{\theta}{2}\cos\left(\psi - \frac{\phi}{2}\right), \qquad x^4 = 2\sqrt{r}\sin\frac{\theta}{2}\sin\left(\psi - \frac{\phi}{2}\right). \quad \text{(B.3)}$$

The metric (B.1) has a global $U(1)_L \times SU(2)_R$ symmetry. The $SU(2)_R$ symmetry refers to the usual rotation group of three dimensional space and will not be of interest to us. The $U(1)_L$ symmetry acts as

$$\psi \to \psi + \epsilon, \quad \text{(B.4)}$$

with no action on any of the other coordinates. From the point of view of an asymptotic observer this is just a translation along the compact circle $\widetilde{S}^1$ parametrized by ψ, and the corresponding conserved charge is the quantum number j_0. On the other hand using (B.3) we see that near the origin the ψ translation acts as simultaneous rotation along the 1-2 and 3-4 planes. Thus near the origin the contribution to the j_0 charge can be identified as the sum of the angular momentum in the 1-2 and 3-4 planes.

Let us now analyze the transformation laws of the fermion fields under $U(1)_L$ transformation. Since the fermions transform under the tangent space group, we need to find the compensating tangent space rotation that accompanies the global $U(1)_L$ transformation. Let $SO(4)^T = SU(2)_L^T \times SU(2)_R^T$ be the tangent space rotation group of the Taub-NUT space, – this needs to be distinguished from the global symmetry group described earlier. The Taub-NUT space is known to have $SU(2)$ holonomy. We shall choose our convention such that the holonomy is in the $SU(2)_L^T$. To see what this means we use the fact that for a suitable choice of the tangent space basis vectors (e^0, e^1, e^2, e^3) the spin connection $\omega^{ab} = \omega_\mu^{ab} dx^\mu$ associated with the metric (B.1) is given by [38]

$$\omega^{01} = -\left(r + \frac{R}{2}\right)(r + R)^{-1}\sigma_x, \qquad \omega^{02} = -\left(r + \frac{R}{2}\right)(r + R)^{-1}\sigma_y,$$

$$\omega^{03} = -\frac{R^2}{2}(r + R)^{-2}\sigma_z, \qquad \omega^{23} = -\frac{R}{2}(r + R)^{-1}\sigma_x,$$

$$\omega^{31} = -\frac{R}{2}(r + R)^{-1}\sigma_y, \qquad \omega^{12} = \left(\frac{R^2}{2}(r + R)^{-2} - 1\right)\sigma_z, \quad \text{(B.5)}$$

where σ_x, σ_y and σ_z are a set of one-forms

$$
\begin{aligned}
\sigma_x &= \cos{(2\psi)}\,d\theta + \sin{(2\psi)}\sin\theta d\phi\,, \\
\sigma_y &= -\sin{(2\psi)}\,d\theta + \cos{(2\psi)}\sin\theta d\phi\,, \\
\sigma_z &= \cos\theta d\phi + 2d\psi\,.
\end{aligned}
\tag{B.6}
$$

Thus near the origin $r = 0$ we have

$$
\omega^{01} = \omega^{23} = -\frac{1}{2}\sigma_x, \qquad \omega^{02} = \omega^{31} = -\frac{1}{2}\sigma_y, \qquad \omega^{03} = \omega^{12} = -\frac{1}{2}\sigma_z\,.
\tag{B.7}
$$

If Σ^{ab} denotes the generator of the tangent space group then we see from (B.7) that near the origin the holonomy group is generated by the combinations $\Sigma^{01} + \Sigma^{23}$, $\Sigma^{02} + \Sigma^{31}$ and $\Sigma^{03} + \Sigma^{12}$. These generate an SU(2) group, – by our convention this describes the tangent space $\mathrm{SU}(2)_L^T$ group near the origin. The precise relation between the Σ^{ab}'s and the generators T^1, T^2, T^3 of $\mathrm{SU}(2)_L^T$ are

$$
T^1 = \frac{1}{2}(\Sigma^{01} + \Sigma^{23}), \qquad T^2 = \frac{1}{2}(\Sigma^{02} + \Sigma^{31}), \qquad T^3 = \frac{1}{2}(\Sigma^{03} + \Sigma^{12})\,,
\tag{B.8}
$$

so that we have

$$
\omega^{ab}\Sigma^{ab} = -\sigma_x T^1 - \sigma_y T^2 - \sigma_z T^3\,.
\tag{B.9}
$$

Using (B.6) we can express this as

$$
\omega^{ab}\Sigma^{ab} = -iU_0\,d\,U_0^{-1}\,,
\tag{B.10}
$$

where

$$
U_0 = e^{2i\psi T_3}e^{i\theta T_1}e^{i\phi T_3}\,.
\tag{B.11}
$$

Now let us consider the effect of a global $\mathrm{U}(1)_L$ rotation $\psi \to \psi + \epsilon$. Using (B.10), (B.11) we see that this induces a transformation

$$
\omega^{ab}\Sigma^{ab} \to e^{2i\epsilon T_3}\omega^{ab}\Sigma^{ab}e^{-2i\epsilon T_3}\,.
\tag{B.12}
$$

Since T_3 is a generator of $\mathrm{SU}(2)_L^T$ we see that in order to preserve the action a global $\mathrm{U}(1)_L$ rotation must be accompanied by a compensating $\mathrm{SU}(2)_L^T$ transformation. In particular the fermion fields χ must transform as

$$
\chi \to e^{2i\epsilon T_3}\chi\,,
\tag{B.13}
$$

where T_3 is to be taken in the representation of $\mathrm{SU}(2)_L^T$ in which the fermions transform.

Now we know that the fermions transform in the $(1,2) + (2,1)$ representation of the tangent space $\mathrm{SU}(2)_L^T \times \mathrm{SU}(2)_R^T$ group. Thus half of the fermions are neutral under $\mathrm{SU}(2)_L^T$ and hence also under the global $\mathrm{U}(1)_L$. On the other hand since the holonomy is in $\mathrm{SU}(2)_L^T$, there is no coupling of these fermions to the spin connection and they behave as free fermions. The other half of the fermions, carrying non-trivial $\mathrm{SU}(2)_L^T$ quantum numbers, are interacting and do transform under the global $\mathrm{U}(1)_L$ group. Furthermore, since type IIB string theory on K3 is a chiral theory, the world-sheet chirality of these fermions is

correlated with the chirality under the tangent space rotation group. In particular, the $SU(2)_L^T$ singlet free fermions are left-moving on the D1-D5-brane world-sheet, and the $SU(2)_R^T$ singlet interacting fermions are right-moving on the world-sheet.[7]

This shows that the resulting (1+1) dimensional world-volume theory on the D1-D5 system is described by a set of four free left-moving $U(1)_L$ invariant fermion fields, together with an interacting theory of four bosons and four right-moving $U(1)_L$ non-invariant fermions. Let us first calculate the contribution to the partition function due to the free left-moving fermions. Since these fermons do not carry any j_0 charge, their contribution is given by:

$$
\begin{aligned}
Z_{\text{free}}(\hat{\rho}) &\equiv \text{Tr}_{\text{free left-moving fermions}}((-1)^{F+\bar{F}}e^{2\pi i \hat{\rho} l_0 + 2\pi i \hat{v} j_0}), \\
&= 4\prod_{n=1}^{\infty}(1 - e^{2\pi i n N \hat{\rho}})^4,
\end{aligned}
\tag{B.14}
$$

taking into account the identification $l_0 = (L_0 - \bar{L}_0)/N$. The factor of 4 comes from the quantization of the free fermion zero modes. The latter in turn can be interpreted as due to the four broken supersymmetries of the D1-D5-system on $K3\times$Taub-NUT space.

Now we turn to the interacting part of the theory. Since we are computing an index we can assume that it does not depend on continuously varying parameters. Let us take the size R of the Taub-NUT space to be large so that the metric is almost flat and in a local region of the Taub-NUT space the world-volume theory of the D1-D5 system is almost free. In this case we should be able to compute the contribution to the non-zero mode bosonic and fermonic oscillators by placing the D1-D5 system at any point in Taub-NUT space, – say at the origin. The contribution from the zero modes however is sensitive to the global geometry of the Taub-NUT space and should be computed separately.

Since supersymmetry acts on the right-moving bosons and fermions, in order to get a BPS state the right-moving bosonic and fermionic oscillators must be in their ground state. Thus as far as the contribution due to the non-zero mode oscillators are concerned, we only need to examine the effect of left-moving bosonic oscillators carrying $L_0 = l_0/N$, $\bar{L}_0 = 0$ and angular momentum j_0. From (B.3) we see that a translation of ψ induces simultaneous rotation in the $x^1 - x^2$ and $x^3 - x^4$ plane. Hence we need to compute the partition function of free left-moving bosons carrying total angular momentum j_0 in the $x^1 - x^2$ and $x^3 - x^4$ plane. This is easily done using the result of [39]. The answer is

$$
\begin{aligned}
Z_{\text{osc}}(\hat{\rho}, \hat{v}) &\equiv \text{Tr}_{\text{oscillators}}((-1)^{F+\bar{F}}e^{2\pi i \hat{\rho} l_0 + 2\pi i \hat{v} j_0}) \\
&= \prod_{n=1}^{\infty}\frac{1}{(1 - e^{2\pi i n N \hat{\rho} + 2\pi i \hat{v}})^2(1 - e^{2\pi i n N \hat{\rho} - 2\pi i \hat{v}})^2}.
\end{aligned}
\tag{B.15}
$$

The result can be understood as follows. The bosonic oscillators in the $1 - 2$ plane and the $3 - 4$ plane can be split into complex conjugate pairs which carry opposite units of angular

---

[7]This can be seen as follows. Since the bosons are interacting, the free fermions have no bosonic superpartner, and hence they do not transform under the unbroken supersymmetry of the D1-D5 system in $K3\times$Taub-NUT space. Thus they must be left-moving. On the other hand the $SU(2)_R^T$ singlet interacting fermions are the superpartners of the interacting bosonic fields and hence must be right-moving on the world-sheet.

momentum. Therefore two of the bosonic oscillators carry $+1$ unit of angular momentum and the other two carry -1 unit of angular momentum. The partition function weighted with the angular momentum is then given by (B.15).

Finally we turn to the contribution Z_z to the partition function from the bosonic and fermionic zero modes of the interacting part of the theory. Since the intercting theory has four bosonic and four fermionic fields, the dynamics is that of a superparticle with four bosonic and four fermionic coordinates moving in Taub-NUT space. Under the holonomy group $SU(2)_L^T$ of the Taub-NUT space both the bosons and the fermions transform in a pair of spinor representations. This system is described by an $N = 4$ supersymmetric quantum mechanics. Thus in order to look for BPS states of the D1-D5 system we need to look for supersymmetric ground states of this quantum mechanics.

So far we have been working at a special point in the moduli space of the CHL string theory where the circles S^1 and $\widetilde{S}^1$ are orthonormal in the asymptotic geometry. In this case the BPS mass of the D1-D5-Kaluza-Klein monopole system is equal to the sum of the BPS masses of the D1-D5 system and the Kaluza-Klein monopole. As a result there is no potential term in the D1-D5 world-volume action and analysis of bound states is difficult. But this is not a generic situation. Once we switch on a component of the metric that mixes S^1 and $\widetilde{S}^1$ the BPS mass of the D1-D5-Kaluza-Klein monopole system is less than the sum of the BPS masses of the D1-D5 system and the Kaluza-Klein monopole. In this case the D1-D5 system in the presence of the monopole has a potential and the system is easier to analyze. On the other hand the analysis of the dynamics of non-zero modes will not be affected by this modification since we are computing an index, – hence our results for Z_{free} and Z_{osc} should remain unchanged.

The mixing between S^1 and $\widetilde{S}^1$ can be achieved by replacing the $d\psi$ term in the expression for the metric given in (B.1) by $d\psi + a\,dy$ where y is the coordinate along S^1 and a is a small deformation parameter. This clearly remains a solution of the equations of motion but gives an r dependent contribution to the yy component of the metric:

$$\Delta g_{yy} = 4\,a^2\,R^2\left(1 + \frac{R}{r}\right)^{-1}. \tag{B.16}$$

As a result the tension of the D1-D5 system, being proportional to $\sqrt{g_{yy}}$, acquires an r-dependent contribution proportional to

$$a^2 R^2 \left(1 + \frac{R}{r}\right)^{-1} \tag{B.17}$$

to first order in a^2. Supersymmetrization of this term gives rise to other fermionic terms.

Thus we have to analyze the dynamics of a superparticle with $\mathcal{N} = 4$ supersymmetry moving in Taub-NUT space under a potential proportional to (B.17). This is precisely the problem analyzed in [40, 41] in a different context. The result of this analysis can be summarized as follows. Depending on the sign of the deformation parameter a we have supersymmetric bound states for $j_0 > 0$ or $j_0 < 0$, where j_0 is the momentum conjugate to the coordinate ψ. In the weak coupling limit the number of bound states for a given value

of j_0 is given by $|j_0|$. If for definiteness we choose the sign of a such that we get bound states for positive j_0, then this gives the zero mode partition function

$$Z_z(\hat{v}) = Tr_{\text{zero modes}}((-1)^{F+\bar{F}} e^{2\pi i \hat{v} j_0}) = \sum_{j_0=1}^{\infty} j_0 \, e^{2\pi i \hat{v} j_0} = \frac{e^{2\pi i \hat{v}}}{(1 - e^{2\pi i \hat{v}})^2} \, . \tag{B.18}$$

Since this is invariant under $\hat{v} \to -\hat{v}$ we shall get the same answer if we had chosen to work with the opposite sign of a that produces bound states with negative j_0.

Finally putting all the ingredients together the partition function of states associated with the centre of mass motion of the D1-D5 system in Taub-NUT space is given by

$$\sum_{l_0, j_0} d_{CM}(l_0, j_0) e^{2\pi i l_0 \hat{\rho} + 2\pi i j_0 \hat{v}} = Z_{\text{free}}(\hat{\rho}) Z_{\text{osc}}(\hat{\rho}, \hat{v}) Z_z(\hat{v}) \tag{B.19}$$

$$= 4 e^{-2\pi i \hat{v}} (1 - e^{-2\pi i \hat{v}})^{-2}$$

$$\times \prod_{n=1}^{\infty} \{(1 - e^{2\pi i n N \hat{\rho}})^4 (1 - e^{2\pi i n N \hat{\rho} + 2\pi i \hat{v}})^{-2} (1 - e^{2\pi i n N \hat{\rho} - 2\pi i \hat{v}})^{-2}\}.$$

C. Proof of $\widetilde{\Phi}_k(T, U, V) = \widetilde{\Phi}_k(NU, T/N, V)$

The Siegel modular form which appears in the expression for the dyon spectrum of the $\mathbb{Z}_N$ CHL model is given by [8]

$$\widetilde{\Phi}_k(U, T, V) = -(i\sqrt{N})^{-k-2} \exp\left(2\pi i \left(\frac{1}{N} T + U + V\right)\right)$$

$$\prod_{r=0}^{N-1} \prod_{\substack{l, b \in \mathbb{Z}, k' \in \mathbb{Z} + \frac{r}{N} \\ k', l, b > 0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2} \sum_{s=0}^{N-1} e^{-2\pi i l s / N} c^{(r,s)}(4lk' - b^2)}$$

$$\prod_{r=0}^{N-1} \prod_{\substack{l, b \in \mathbb{Z}, k' \in \mathbb{Z} - \frac{r}{N} \\ k', l, b > 0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2} \sum_{s=0}^{N-1} e^{2\pi i l s / N} c^{(r,s)}(4lk' - b^2)}$$

$$\tag{C.1}$$

where $(k', l, b) > 0$ means $k' > 0, l \geq 0, b \in \mathbb{Z}$ or $k' = 0, l > 0, b \in \mathbb{Z}$ or $k' = 0, l = 0, b < 0$. The coefficients $c^{(r,s)}$ have been defined in (3.9)-(3.13). From this definition it follows that $c^{(r,s)}(4n - b^2) = c^{(-r,-s)}(4n - b^2)$. Hence the two products in (C.1) give identical results and we may rewrite this as

$$\widetilde{\Phi}_k(U, T, V) = -(i\sqrt{N})^{-k-2} \exp\left(2\pi i \left(\frac{1}{N} T + U + V\right)\right)$$

$$\prod_{r=0}^{N-1} \prod_{\substack{l, b \in \mathbb{Z}, k' \in \mathbb{Z} + \frac{r}{N} \\ k', l, b > 0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\sum_{s=0}^{N-1} e^{-2\pi i l s / N} c^{(r,s)}(4lk' - b^2)}$$

$$\tag{C.2}$$

For $N = 1$ this reduces to the familiar weight 10 cusp form of the modular group of genus two Riemann surfaces [42–44].

If we define $j = Nk'$ in (C.2), then in the above product $r = j \bmod N$, and we get

$$\widetilde{\Phi}_k(U, T, V) = -(i\sqrt{N})^{-k-2} \exp\left(2\pi i \left(\frac{1}{N} T + U + V\right)\right)$$
$$\prod_{\substack{l,b \in \ \mathbb{Z}, j \in \ \mathbb{Z} \\ j,l,b > 0}} \left\{1 - \exp(2\pi i(jT/N + lU + bV))\right\}^{\sum_{s=0}^{N-1} e^{-2\pi i ls/N} c^{(j,s)}(4lj/N - b^2)}$$
(C.3)

Substituting $U \to T/N$, $T \to UN$ in this product and exchanging the dummy indices j and l we get:

$$\widetilde{\Phi}_k\big(NU, T/N, V\big) = -(i\sqrt{N})^{-k-2} \exp\left(2\pi i \left(\frac{1}{N} T + U + V\right)\right)$$
$$\prod_{\substack{l,b \in \ \mathbb{Z}, j \in \ \mathbb{Z} \\ j,l,b > 0}} \left\{1 - \exp(2\pi i(jT/N + lU + bV))\right\}^{\sum_{s=0}^{N-1} e^{-2\pi i js/N} c^{(l,s)}(4lj/N - b^2)}$$
(C.4)

Now using (3.9) one can check explicitly that[8]

$$\sum_{s=0}^{N-1} e^{-2\pi i ls/N} F^{(j,s)}(\tau, z) = \sum_{s=0}^{N-1} e^{-2\pi i js/N} F^{(l,s)}(\tau, z),$$
(C.5)

and hence from (3.13)

$$\sum_{s=0}^{N-1} e^{-2\pi i ls/N} c^{(j,s)}(4n - b^2) = \sum_{s=0}^{N-1} e^{-2\pi i js/N} c^{(l,s)}(4n - b^2).$$
(C.6)

Eq. (C.6) shows that the right hand sides of (C.3) and (C.4) are identical. Hence

$$\widetilde{\Phi}_k(T, U, V) = \widetilde{\Phi}_k(NU, T/N, V).$$
(C.7)

D. The D1-D5 system and SCFT with target space $S^P K3/\ \mathbb{Z}_N$

In this appendix we evaluate the degeneracy of the low lying states of the D1-D5 system described in sections 2 and 3 for $Q_5 \geq 1$. Before the $\mathbb{Z}_N$ orbifolding the dynamics of the relative motion of Q_5 D5-branes wrapped on $K3 \times S^1$ and Q_1 D1-branes wrapped on S^1 is captured by the $\mathcal{N} = (4, 4)$ superconformal σ-model with the symmetric product of $Q_5(Q_1 - Q_5) + 1$ copies of $K3$ as the target space [24]. We shall denote this target space by $S^P K3 \equiv (K3)^P/S_P$, where S_P refers to the permutation group of P elements. The

---

[8]For this we need to use the relation $\sum_{k=0}^{N-1} E_N\left(\frac{\tau+k}{N}\right) = N E_N(\tau)$. This follows from the Fourier expansion of $E_N(\tau)$ given in (3.12).

world-sheet coordinate σ of this conformal field theory is identified with the coordinate along S^1. We shall first review various aspects of the superconformal field theory with target space $(K3)^P/S_P$, and then discuss the effect of the $\mathbb{Z}_N$ projection that is required to describe a D1-D5-brane configuration on the CHL orbifold.

Let g be an element of S_P and $[g]$ denote the conjugacy class of g. Then the Hilbert space of the SCFT with target space $(K3)^P/S_P$ decomposes into a direct sum of twisted sectors labelled by the conjugacy classes of S_P:

$$\mathcal{H} = \oplus_{[g]} \mathcal{H}_g^{(\mathcal{C}_g)} \tag{D.1}$$

where $\mathcal{C}_g$ denotes the centralizer of $[g]$ and $\mathcal{H}_g^{(\mathcal{C}_g)}$ refers to the Hilbert space in the g twisted sector projected by $\mathcal{C}_g$. The conjugacy classes of S_P may be labelled as

$$[g] = (1)^{P_1}(2)^{P_2} \cdots (s)^{P_s} \tag{D.2}$$

where (w) denotes cyclic permutation of w elements and P_w is the number of copies of (w) in g. Thus these conjugacy classes are characterized by partitions P_w of P such that

$$\sum_w w P_w = P. \tag{D.3}$$

The centralizer $\mathcal{C}_g$ of the conjugacy class $[g]$ given in (D.2) is given by

$$\mathcal{C}_g = S_{P_1} \times (S_{P_2} \times \mathbb{Z}_2^{P_2}) \times \cdots \times (S_{P_s} \times \mathbb{Z}_s^{P_s}). \tag{D.4}$$

If we denote by $\mathcal{H}_w$ the Hilbert space of states twisted by the generator ω of the $\mathbb{Z}_w$ group of cyclic permutation of w elements, and projected by the same $\mathbb{Z}_w$ group, then (D.4) shows that for the conjugacy class $[g]$ given in (D.2)

$$\mathcal{H}_g^{(\mathcal{C}_g)} = \otimes_{w>0} S^{P_w} \mathcal{H}_w \tag{D.5}$$

Consider first the Hilbert space $\mathcal{H}_w$. This twisted sector is represented by the Hilbert space of the sigma model of w coordinate fields $X_i(\sigma) \in K3$ with the cyclic boundary condition

$$X_i(\sigma + 2\pi) = \omega X_i(\sigma) = X_{i+1}(\sigma), \quad i \in (1, \ldots, w), \tag{D.6}$$

where ω acts by $\omega : X_i \to X_{i+1}$. Therefore the w coordinate fields can be glued together as a single field but in the interval $0 \leq \sigma \leq 2\pi w$, moving in the target space K3. Thus we now have a string of length $2\pi w$, – commonly known as the long string, – moving in K3. The natural unit of momentum on this string is $1/w$, and hence if a state of this string carries physical momentum $-l$ along S^1, it corresponds to $L_0 - \bar{L}_0$ eigenvalue lw.

At this stage it is worth pointing out that whereas for $Q_5 = 1$ the quantum number w can be identified with the winding charge of the D-string, this is not so for $Q_5 > 1$. Thus we should not regard the long string as a D-string, – rather it provides some effective description of the dynamics.

Once we know the spectrum of $\mathcal{H}_w$, the full spectrum of the CFT of the D1-D5 system is obtained by taking the direct product of the spectrum of $\mathcal{H}_w$'s and then carrying out appropriate symmetrization described in (D.5).

We now turn to the effect of the $\mathbb{Z}_N$ projection that is required in order to get a state of the D1-D5 system in the $\mathbb{Z}_N$ CHL model. For this we need to pick a $\mathbb{Z}_N$ invariant state from each of the $\mathcal{H}_w$. Thus we need to first study the effect of the $\mathbb{Z}_N$ projection in each $\mathcal{H}_w$ and then take the direct product of these $\mathbb{Z}_N$ invariant subspaces followed by appropriate symmetrization. In particular if we denote by $\mathcal{H}'_w$ the subspace of $\mathbb{Z}_N$ invariant states in $\mathcal{H}_w$, then the full $\mathbb{Z}_N$ invariant Hilbert space of the D1-D5 system will be given by

$$\otimes_{w>0} S^{P_w} \mathcal{H}'_w. \tag{D.7}$$

The $\mathbb{Z}_N$ projection on a single long string is implemented as follows. Since the long string has length $2\pi w$, we need to impose periodic boundary condition on this string with period $2\pi w/N$. Since under a translation by $2\pi w/N$ on the string the physical coordinate along S^1 gets shifted by $2\pi r/N$ where $r = w \bmod N$, under such a shift the coordinates of the string along K3 must be transformed by $\widetilde{g}^r$ where $\widetilde{g}$ is the generator of the $\mathbb{Z}_N$ action on K3. Thus if $\widetilde{\mathbb{Z}}_N$ denotes the group generated by $\widetilde{g}$, we can view the long string as a closed string of length $2\pi w/N$ with target space $K3/\widetilde{\mathbb{Z}}_N$, belonging to a sector of this SCFT twisted by $\widetilde{g}^r$. Since the natural unit of momentum along the string is now N/w, a physical momentum $-l$ along S^1 would correspond to $L_0 - \bar{L}_0$ eigenvalue of lw/N in the SCFT describing the dynamics of the long string.

Now from (3.14) we know that for an SCFT with target space $K3/\widetilde{\mathbb{Z}}_N$

$$\frac{1}{N} Tr_{RR;\widetilde{g}^r} (\widetilde{g}^s (-1)^{F+\bar{F}} \delta_{NL_0, lw} \delta_{\mathcal{J},j}) = c^{(r,s)}(4lw/N - j^2). \tag{D.8}$$

Here $\mathcal{J}$ is the angular momentum operator. Also the projection operator for $\mathbb{Z}_N$ invariant states with physical momentum $-l$ along S^1 is:

$$\frac{1}{N} \sum_s e^{-2\pi i l s} \widetilde{g}^s. \tag{D.9}$$

Hence the total number of bosonic minus fermionic states in the single long string Hilbert space, carrying momentum $-l$ along S^1 and angular momentum j is given by:

$$\frac{1}{N} \sum_s e^{-2\pi i l s} Tr_{RR;\widetilde{g}^r} (\widetilde{g}^s (-1)^{F+\bar{F}} \delta_{NL_0, lw} \delta_{\mathcal{J},j}) = \sum_s e^{-2\pi i l s} c^{(r,s)}(4lw/N - j^2) \equiv n(w,l,j). \tag{D.10}$$

According to (D.7) the next step is the evaluation of the partition function for the symmetrized tensor products of the Hilbert spaces $\mathcal{H}'_w$. For this we use the following formula from [45]. If $d_{sym}(P_w, w, L, J)$ denotes the number of bosonic minus fermionic states in $S^{P_w} \mathcal{H}'_w$ carrying total momentum $-L$ along S^1 and total angular momentum J, then

$$\sum_{P_w=0}^{\infty} \sum_{L,J} d_{sym}(P_w, w, L, J) e^{2\pi i L\hat{\rho} + 2\pi i J\hat{v} + 2\pi i y P_w} = \prod_{\substack{l,j \in \mathbb{Z} \\ l \geq 0}} \left(1 - e^{2\pi i y + 2\pi i l\hat{\rho} + 2\pi i j\hat{v}}\right)^{-n(w,l,j)}. \tag{D.11}$$

The proof of this given in [45] holds provided that the coeficients $n(w, l, j)$ are integers. Therefore to apply this formula for the Hilbert space $\mathcal{H}'_w$ we need

$$n(w, l, j) = \sum_{s=0}^{N-1} c^{(r,s)}(4lw/N - j^2)e^{-2\pi ils/N} \tag{D.12}$$

to be an integer. This can be explicity verified using the formulae given in (3.9)-(3.12).

Using the identity in (D.11) we can evaluate the generating function for the bosonic minus fermionic states for the relative dynamics of the D1-D5 system. Eq. (D.7) shows that all we need to do is to take the product over w of the right hand side of (D.11). More specifically, if $h'(P, n, J)$ denotes the total number of bosonic minus fermionic states carrying total string length $2\pi P/N = 2\pi \sum_{w>0} wP_w/N$ (counting a single string with quantum number w to have length $2\pi w/N$), total momentum $-n$ along S^1 and total angular momentum J, then we have

$$
\begin{aligned}
&\sum_{P,n,J} h'(P, n, J)e^{2\pi i(\hat{\rho}n + \hat{\sigma}(P-1)/N + \hat{v}J)} \\
&= e^{-2\pi i\hat{\sigma}/N} \prod_{\substack{w \in \mathbb{Z} \\ w>0}} \prod_{\substack{l,j \in \mathbb{Z} \\ l \geq 0}} \left(1 - e^{2\pi i\hat{\sigma}w/N + 2\pi il\hat{\rho} + 2\pi ij\hat{v}}\right)^{-n(w,l,j)} \\
&= e^{-2\pi i\hat{\sigma}/N} \prod_{r=0}^{N-1} \prod_{\substack{p' \in \mathbb{Z} + \frac{r}{N}, l,j \in \mathbb{Z} \\ p'>0, l \geq 0}} (1 - e^{2\pi i(p'\hat{\sigma} + l\hat{\rho} + j\hat{v})})^{-\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(r,s)}(4p'l - j^2)} .
\end{aligned}
\tag{D.13}
$$

We have multiplied $e^{-2\pi i\hat{\sigma}/N}$ on both sides of the above equation to absorb the shift of unity in the value of P. Also in arriving at the last expression in (D.13) we have defined $p' = w/N$. Physically the quantum number P corresponds to $P = Q_5(Q_1 - Q_5) + 1$.

Let us now evaluate the full degeneracy $h(P, n, J)$ of the D1-D5 system and the Taub-NUT space, counting each supermultiplet as one state. To do this we need to put in the contribution from the excitation modes of the Kaluza-Klein monopole and the center of mass motion of the D1-D5 system. These have already been computed in appendices A, B. Putting together all the results we get

$$
\begin{aligned}
f(\hat{\rho}, \hat{\sigma}, \hat{v}) &\equiv \sum_{P,n,J} h(P, n, J)e^{2\pi i(\hat{\rho}n + \hat{\sigma}(P-1)/N + \hat{v}J)} \\
&= \frac{1}{64} e^{-2\pi i\hat{\sigma}/N} \prod_{r=0}^{N-1} \prod_{\substack{p' \in \mathbb{Z} + \frac{r}{N}, l,j \in \mathbb{Z} \\ p'>0, l \geq 0}} (1 - e^{2\pi i(p'\hat{\sigma} + l\hat{\rho} + j\hat{v})})^{-\sum_{s=0}^{N-1} e^{-2\pi isl/N} c^{(r,s)}(4p'l - j^2)} \\
&\quad \left(\sum_{l_0, j_0} d_{CM}(l_0, j_0)e^{2\pi i(l_0\hat{\rho} + j_0\hat{v})}\right) \left(\sum_{l'_0} d_{KK}(l'_0)e^{2\pi il'_0\hat{\rho}}\right)
\end{aligned}
\tag{D.14}
$$

The factor of 1/64 in this expression accounts for the fact that each supermultiplet contains

64 states. Using (3.2), (3.3) one can reduce the above equation to

$$f(\hat{\rho}, \hat{\sigma}, \hat{v}) = e^{-2\pi i(\hat{\sigma}/N + \hat{\rho} + \hat{v})} \prod_{r=0}^{N-1} \prod_{\substack{p' = \mathbb{Z} + \frac{r}{N}, l \in \mathbb{Z}, j \\ p' \geq 0, l \geq 0}} (1 - e^{2\pi i(p'\hat{\sigma} + l\hat{\rho} + j\hat{v})})^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(r,s)}(4p'l - j^2)}$$

(D.15)

where the product over j runs over integer values for $(p', l) \neq (0,0)$ and runs over only positive integers (or negative integers) when $p' = l = 0$. Note that the $p' = 0$ terms in the above equation arise from the terms involving $d_{CM}(l_0, j_0)$ and $d_{KK}(l'_0)$. Now comparing the right hand side of this equation with (C.2) we can write

$$f(\hat{\rho}, \hat{\sigma}, \hat{v}) = -\frac{(i\sqrt{N})^{-k-2}}{\Phi_k(\hat{\rho}, \hat{\sigma}, \hat{v})}$$

(D.16)

From this point onward the analyis of the degeneracy of the D1-D5 system in Taub-NUT space proceeds as in section 3 with $Q_m^2 = 2Q_5(Q_1 - Q_5)$. We have thus extended the analysis of section 3 to the $Q_5 \geq 1$ case.

E. Zeroes and poles of $\widetilde{\Phi}_k$

As has been shown in [8] the Siegel modular form $\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ given in (C.2) satisfies the relation:

$$-2\ln\widetilde{\Phi}_k - 2\ln\bar{\widetilde{\Phi}}_k - 2k\ln\det Im\Omega + \text{constant}$$
$$= \sum_{r,s=0}^{N-1} \sum_{l=0}^{1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_1,m_2,n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z}+\frac{r}{N}, b \in 2\mathbb{Z}+l}} \exp\left[2\pi i\tau(m_1 n_1 + m_2 n_2 + \frac{b^2}{4})\right] \times$$
$$\exp\left(\frac{-\pi\tau_2}{Y} \left|n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2\right|^2\right) e^{2\pi i m_1 s/N} h_l^{(r,s)}(\tau),$$

(E.1)

where

$$\Omega = \begin{pmatrix} \widetilde{\sigma} & \widetilde{v} \\ \widetilde{v} & \widetilde{\rho} \end{pmatrix}, \qquad Y = \det Im\Omega,$$

(E.2)

and

$$h_0^{(r,s)}(\tau) = \frac{1}{2\vartheta_3(2\tau, 2z)} \left(F^{(r,s)}(\tau, z) + F^{(r,s)}(\tau, z + \frac{1}{2})\right),$$
$$h_1^{(r,s)}(\tau) = \frac{1}{2\vartheta_2(2\tau, 2z)} \left(F^{(r,s)}(\tau, z) - F^{(r,s)}(\tau, z + \frac{1}{2})\right).$$

(E.3)

The functions $F^{(r,s)}(\tau, z)$ have been defined in eqs. (3.9). The coefficients $c^{(r,s)}(4n)$ introduced in (3.13) are related to $h_l^{(r,s)}(\tau)$ through the expansion:

$$h_0^{(r,s)}(\tau) = \sum_{n \in \mathbb{Z}/N} c^{(r,s)}(4n)q^n, \qquad h_1^{(r,s)}(\tau) = \sum_{n \in \mathbb{Z}/N - \frac{1}{4}} c^{(r,s)}(4n)q^n, \qquad q \equiv e^{2\pi i\tau}.$$

(E.4)

From eqs. (3.9)-(3.13) it follows that the only non-zero $c^{(r,s)}(4n)$ with negative n are $c^{(r,s)} \times (-1)$ for all N, and $c^{(r,s)}(-1 + \frac{4}{N})$ for $N = 5, 7$. They have the values:

$$c^{(0,s)}(-1) = \frac{2}{N},$$

$$c^{(r,s)}(-1) = 0 \qquad \text{for } r \neq 0 \bmod N,$$

$$c^{(0,s)}(-1 + \frac{4}{N}) = 0,$$

$$c^{(r,rk)}(-1 + \frac{4}{N}) = -\frac{48}{N(N+1)(N-1)} e^{2\pi i k/N}, \qquad \text{for } r \neq 0 \bmod N. \qquad \text{(E.5)}$$

Eq. (E.1) shows that the zeroes and poles of $\widetilde{\Phi}_k$ appear only when the τ integral on the right hand side of this equation diverges from the region near $\tau = i\infty$. Now, if we consider a term proportional to $e^{2\pi i n \tau}$ in the expansion of $h_l^{(r,s)}$, then for large τ_2, the τ_1 integral gives a non-vanishing answer only if

$$n + m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = 0. \qquad \text{(E.6)}$$

Thus after performing the τ_1 integral, the only τ_2 dependence of the integrand in the large τ_2 region comes from the

$$-\frac{\pi \tau_2}{Y} \left| n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \right|^2 \qquad \text{(E.7)}$$

factor in the exponent. Since this is negative, the only way the integral can diverge from the large τ_2 region is if this vanishes:

$$n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 = 0, \qquad \text{(E.8)}$$

for some m_1, m_2, n_1, n_2, b appearing in the sum in (E.1).

Now we have the identity

$$m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{2}(p_L^2 - p_R^2), \qquad \text{(E.9)}$$

where

$$p_R^2 = \frac{1}{2Y} \left| n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \right|^2,$$

$$p_L^2 = \frac{1}{2Y} \left\{ m_2 + n_2(\widetilde{\sigma}_2\widetilde{\rho}_2 + \widetilde{\sigma}_1\widetilde{\rho}_1 - \widetilde{v}_1^2 - \widetilde{v}_2^2) - m_1\widetilde{\rho}_1 + n_1\widetilde{\sigma}_1 + b\widetilde{v}_1 \right\}^2$$

$$+ \frac{1}{2Y} \left\{ n_2 \left(\widetilde{\sigma}_1\widetilde{\rho}_1 - \widetilde{\sigma}_1\widetilde{\rho}_2 + 2\widetilde{v}_1\widetilde{v}_2 - \frac{2\widetilde{v}_2^2\widetilde{\rho}_1}{\widetilde{\rho}_2} \right) + m_1\widetilde{\rho}_2 + n_1 \left(\widetilde{\sigma}_2 - \frac{2\widetilde{v}_2^2}{\widetilde{\rho}_2} \right) - b\widetilde{v}_2 \right\}^2$$

$$+ 2 \left\{ \frac{b}{2} + n_1\frac{\widetilde{v}_2}{\widetilde{\rho}_2} - n_2\widetilde{v}_1 + n_2\frac{\widetilde{v}_2\widetilde{\rho}_1}{\widetilde{\rho}_2} \right\}^2. \qquad \text{(E.10)}$$

Since p_L^2 is positive semi-definite, and since p_R^2 vanishes when (E.8) holds, (E.9) shows that we must have

$$m_1 n_1 + m_2 n_2 + \frac{b^2}{4} \geq 0. \qquad \text{(E.11)}$$

Furthermore the equality sign holds only when p_L^2 also vanishes. This requires $m_1 = m_2 = n_1 = n_2 = b = 0$. The corresponding divergence is present for all $\widetilde{\sigma}$, $\widetilde{\rho}$, $\widetilde{v}$ and is removed by a subtraction term [8]. Thus the divergences which depend on $\widetilde{\sigma}$, $\widetilde{\rho}$, $\widetilde{v}$ come from those values of m_i, n_i, b which satisfy (E.8) and for which

$$m_1 n_1 + m_2 n_2 + \frac{b^2}{4} > 0 \,. \tag{E.12}$$

This, together with eq. (E.6), now show that we must have

$$n < 0 \,. \tag{E.13}$$

In other words the only terms in the expansion of $h_l^{(r,s)}$ responsible for a divergent contribution to the integral (E.1) are the ones involving negative powers of q. From eqs. (E.4), (E.5) it now follows that for all N there is a divergent contribution to (E.1) from the $n = -1/4$ term, i.e. the term

$$c^{(r,s)}(-1)q^{-1/4} \,, \tag{E.14}$$

in $h_1^{(r,s)}$. On the other hand for $N = 5, 7$ there are additional divergences from $n = -\frac{1}{4} + \frac{1}{N}$, i.e. the term

$$c^{(r,s)}(-1 + \frac{4}{N})q^{-1/4+1/N} \,, \tag{E.15}$$

in $h_1^{(r,s)}$. Since the subscript of h is odd in both cases, we must have b odd.

First consider the contribution from the $c^{(r,s)}(-1)$ term. (E.6) now gives

$$m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4} \,. \tag{E.16}$$

After estimating the τ_2 integral in (E.1) for $n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \simeq 0$, one easily finds that the divergent contribution is given by

$$-2 \sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c^{(r,s)}(-1) \ln \left| n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \right|^2 \,,$$
$$r = n_1 N \bmod N, \, b = 1 \bmod 2 \,, \tag{E.17}$$

where we have included a factor of 2 due to the fact that the lattice vectors $(\vec{m}, \vec{n}, b)$ and $(-\vec{m}, -\vec{n}, -b)$ give identical divergent contribution. Thus near this region $\widetilde{\Phi}_k$ behaves as

$$\widetilde{\Phi}_k \sim \left(n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \right)^{\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c^{(r,s)}(-1)} \,. \tag{E.18}$$

Using (E.5) we can compute the sum in the exponent. In particular r must vanish for $c^{(r,s)}(-1)$ to be non-zero, and hence n_1 must be an integer. Furthermore, substituting $c^{(0,s)}(-1) = 2/N$ in (E.18) we see that the sum in the exponent vanishes unless m_1 is an integer multiple of N. The final result after performing the sum is

$$\widetilde{\Phi}_k \sim \left(n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho}m_1 + m_2 \right)^2$$
$$\text{for } m_1 \in N\,\mathbb{Z}, \, n_1 \in \mathbb{Z}, \, b \in 2\,\mathbb{Z}+1, \, m_2, n_2 \in \mathbb{Z} \,,$$
$$\sim 1 \quad \text{otherwise} \,. \tag{E.19}$$

For $N = 5, 7$ we also have divergent contribution to (E.1) from the $c^{(r,s)}(-1 + \frac{4}{N})$ term. In this case (E.6) gives

$$m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4} - \frac{1}{N} \,. \tag{E.20}$$

The divergent contribution takes the form

$$-2 \sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c^{(r,s)}(-1 + \frac{4}{N}) \ln \left| n_2 (\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1 \widetilde{\sigma} - \widetilde{\rho} m_1 + m_2 \right|^2 \,,$$
$$r = n_1 N \bmod N, \, b = 1 \bmod 2 \,. \tag{E.21}$$

Thus $\widetilde{\Phi}_k$ behaves as

$$\widetilde{\Phi}_k \sim \left(n_2 (\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1 \widetilde{\sigma} - \widetilde{\rho} m_1 + m_2 \right)^{\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c^{(r,s)}(-1 + \frac{4}{N})} \,. \tag{E.22}$$

We can compute the sum in the exponent by rewriting it as

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c^{(r,s)}(-1 + \frac{4}{N}) = \sum_{k=0}^{N-1} e^{2\pi i m_1 r k/N} c^{(r,rk)}(-1 + \frac{4}{N}) \quad \text{for } r \neq 0 \bmod N \,. \tag{E.23}$$

Using (E.5) we now get

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c^{(r,s)}(-1 + \frac{4}{N}) = \begin{cases} -48/(N^2 - 1) & \text{for } m_1 r = -1 \bmod N \\ 0 & \text{otherwise} \end{cases} \,. \tag{E.24}$$

This gives

$$\widetilde{\Phi}_k \sim \left(n_2 (\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + b\widetilde{v} + n_1 \widetilde{\sigma} - \widetilde{\rho} m_1 + m_2 \right)^{-48/(N^2 - 1)}$$
$$\text{for } m_1 n_1 N = -1 \bmod N, \, n_1 N \neq 0 \bmod N, \, b \in 2\,\mathbb{Z} + 1, \, m_1, m_2, n_2 \in \mathbb{Z},$$
$$\sim 1 \quad \text{otherwise} \,. \tag{E.25}$$

To summarize, for $N = 1, 2, 3, 5, 7$, $\widetilde{\Phi}_k$ has a second order zero at (E.8) for $m_1 = 0$ mod N, $n_1 = 0$ mod 1 and odd b, satisfying (E.16). On the other hand for $N = 5$ and $N = 7$, $\widetilde{\Phi}_k$ also has poles of order $48/(N^2 - 1)$ at (E.8) for $m_1 n_1 N = -1$ mod N, $n_1 N \neq 0$ mod N and odd b, satisfying (E.20). (Note that both for $N = 5$ and $N = 7$, $48/(N^2 - 1)$ is an integer, and hence the singularities are poles and not branch points.)

F. Riemann normal coordinates and duality invariant statistical entropy function

In section 4 we considered $\vec{\eta} = \vec{\tau} - \vec{\tau}_B$ for some fixed base point $\vec{\tau}_B$ as the fundamental field in defining $W_B(\vec{\tau}_B, \vec{J})$ and $\Gamma_B(\vec{\tau}_B, \vec{\chi})$. In this appendix we shall try to generalize this by treating

$$\vec{\xi} = \vec{g}(\vec{\eta}) \tag{F.1}$$

as a fundamental field where $\vec{g}(\vec{\eta})$ is an arbitrary function of $\vec{\eta}$ with a Taylor series expansion starting with the linear terms (i.e. $\vec{g}(\vec{0}) = \vec{0}$). In this case the generating function of $\vec{\xi}$ correlation functions will be given by

$$e^{\widetilde{W}_B(\vec{\tau}_B, \vec{J})} = \int \frac{d^2\eta}{(\tau_{B2} + \eta_2)^2} \, e^{-F(\vec{\tau}_B + \vec{\eta}) + \vec{J} \cdot \vec{g}(\vec{\eta})} \,. \tag{F.2}$$

Thus $\widetilde{W}_B(\vec{\tau}_B, \vec{0}) = S_{stat}$. The corresponding effective action is

$$\widetilde{\Gamma}_B(\vec{\tau}_B, \vec{\psi}) = \vec{J} \cdot \vec{\psi} - \widetilde{W}_B(\vec{\tau}_B, \vec{J}), \qquad \psi_i = \frac{\partial \widetilde{W}_B(\vec{\tau}_B, \vec{J})}{\partial J_i} \,. \tag{F.3}$$

From this definition it follows that

$$J_i = \frac{\partial \widetilde{\Gamma}_B(\vec{\tau}_B, \vec{\psi})}{\partial \psi_i} \,. \tag{F.4}$$

Now suppose $\vec{\tau}^{(0)}$ is a specific value of $\vec{\tau}_B$ for which

$$\left. \frac{\partial \widetilde{\Gamma}_B(\vec{\tau}^{(0)}, \vec{\psi})}{\partial \psi_i} \right|_{\vec{\psi}=0} = 0 \quad \text{i.e.} \quad \left. \frac{\partial \widetilde{W}_B(\vec{\tau}^{(0)}, \vec{J})}{\partial J_i} \right|_{\vec{J}=0} = 0 \,. \tag{F.5}$$

In this case we have $\vec{J} = 0$ for $\vec{\psi} = 0$, and hence

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)}, \vec{0}) = -\widetilde{W}_B(\vec{\tau}^{(0)}, \vec{0}) = -S_{stat} \,. \tag{F.6}$$

We shall now show that $\widetilde{\Gamma}_B(\vec{\tau}_B, \vec{0})$, regarded as a function of $\vec{\tau}_B$, has an extremum at $\vec{\tau}_B = \vec{\tau}^{(0)}$. From (F.3), (F.4) we see that

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{0}) = -\widetilde{W}_B \left(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{J} = \partial \widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{\psi}) / \partial \vec{\psi} \right) \Big|_{\vec{\psi}=\vec{0}} \,. \tag{F.7}$$

Now

$$e^{\widetilde{W}_B(\vec{\tau}_B + \vec{\epsilon}, \vec{J})} = \int \frac{d^2\eta}{(\tau_{B2} + \epsilon_2 + \eta_2)^2} \, e^{-F(\vec{\tau}_B + \vec{\epsilon} + \vec{\eta}) + \vec{J} \cdot \vec{g}(\vec{\eta})} = \int \frac{d^2\eta}{(\tau_{B2} + \eta_2)^2} \, e^{-F(\vec{\tau}_B + \vec{\eta}) + \vec{J} \cdot \vec{g}(\vec{\eta} - \vec{\epsilon})} \,, \tag{F.8}$$

where in the second step we have made a change of variables $\vec{\eta} \to \vec{\eta} - \vec{\epsilon}$. Since $g(\vec{\eta} - \vec{\epsilon}) = g(\vec{\eta}) + O(\vec{\epsilon})$, this shows that

$$\widetilde{W}_B(\vec{\tau}_B + \vec{\epsilon}, \vec{J}) = \widetilde{W}_B(\vec{\tau}_B, \vec{J}) + O(\epsilon_i J_k) \,. \tag{F.9}$$

Using this information in (F.7) we get

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{0}) = -\widetilde{W}_B(\vec{\tau}^{(0)}, \vec{J} = \partial \widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{\psi}) / \partial \vec{\psi})|_{\vec{\psi}=\vec{0}} + O\left(\epsilon_i \frac{\partial \widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{\psi})}{\partial \psi_i} \Big|_{\vec{\psi}=\vec{0}} \right) \,. \tag{F.10}$$

Eq. (F.5) shows that the second term on the right hand side of this equation is of order ϵ^2, and $\vec{J}$ appearing in the argument of the first term is of order ϵ. Expanding the first term

in a Taylor series expansion in $\vec{J}$, and noting that the linear term vanishes due to (F.5), we get

$$\widetilde{\Gamma}_B(\vec{\tau}^{(0)} + \vec{\epsilon}, \vec{0}) = -\widetilde{W}_B(\vec{\tau}^{(0)}, \vec{J} = \vec{0}) + O(\epsilon^2) = \widetilde{\Gamma}_B(\vec{\tau}^{(0)}, \vec{0}) + O(\epsilon^2). \qquad (\text{F.11})$$

Thus

$$\frac{\partial \widetilde{\Gamma}_B(\vec{\tau}, \vec{0})}{\partial \tau_i} = 0 \quad \text{at} \quad \vec{\tau} = \vec{\tau}^{(0)}. \qquad (\text{F.12})$$

Using (F.6) and (F.12) we see that the statistical entropy is given by the value of $-\widetilde{\Gamma}_B(\vec{\tau}, \vec{0})$ at its extremum $\vec{\tau} = \vec{\tau}^{(0)}$. Thus we can identify $-\widetilde{\Gamma}_B(\vec{\tau}, \vec{0})$ as the statistical entropy function. This is computed as the sum of 1PI vacuum amplitudes in the theory with ξ_i regarded as the fundamental fields.

We shall now show that for a suitable choice of the coordinates $\vec{\xi}$, the statistical entropy function $-\widetilde{\Gamma}_B(\vec{\tau}, \vec{0})$ defined this way can be made manifestly duality invariant. This is done by choosing $\vec{\xi}$ as Riemann normal coordinates. For a given base point $\vec{\tau}_B$ we define the coordinate $\vec{\xi}$ for a given point $\vec{\tau}$ in the upper half plane as follows. We introduce the duality invariant metric on the upper half plane

$$ds^2 = (\tau_2)^{-2}(d\tau_1^2 + d\tau_2^2), \qquad (\text{F.13})$$

and draw a geodesic connecting $\vec{\tau}_B$ and $\vec{\tau}$. The coordinate $\vec{\xi}$ corresponding to the point $\vec{\tau}$ is then defined by identifying $|\vec{\xi}|$ as the distance between $\vec{\tau}_B$ and $\vec{\tau}$ along the geodesic and the direction of $\vec{\xi}$ is taken to be the direction of the tangent vector to the geodesic at $\vec{\tau}_B$.[9] Since the metric (F.13) is invariant under a duality transformation, it is clear that if a duality transformation maps $\vec{\tau}_B$ to $\vec{\tau}_B'$ and $\vec{\tau}$ to $\vec{\tau}'$, then the Riemann normal coordinate $\vec{\xi}'$ of $\vec{\tau}'$ with respect to $\vec{\tau}_B'$ will have the property that $|\vec{\xi}'| = |\vec{\xi}|$. Thus $\vec{\xi}$ and $\vec{\xi}'$ are related by a rotation. In order to determine the angle of rotation, we note that under a duality transformation (4.29),

$$d\tau' = (c\tau + d)^{-2}d\tau. \qquad (\text{F.14})$$

Thus

$$\frac{d\tau'}{|d\tau'|} = \frac{|c\tau + d|^2}{(c\tau + d)^2}\frac{d\tau}{|d\tau|}. \qquad (\text{F.15})$$

This shows that a geodesic passing through τ_B gets rotated by a phase $|c\tau_B + d|^2/(c\tau_B + d)^2$ under a duality transformation. Hence

$$\xi' = \frac{|c\tau_B + d|^2}{(c\tau_B + d)^2}\xi = \frac{c\bar{\tau}_B + d}{c\tau_B + d}\xi, \qquad (\text{F.16})$$

where

$$\xi = \xi_1 + i\xi_2, \qquad \xi' = \xi_1' + i\xi_2'. \qquad (\text{F.17})$$

Since for given τ_B the duality transformation acts linearly on $\vec{\xi}$, the corresponding generating function $\widetilde{W}_B(\vec{\tau}_B, \vec{J})$ and the effective action $\widetilde{\Gamma}_B(\vec{\tau}_B, \vec{\psi})$ will be manifestly duality

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[9]Often one uses the convention that the distance along the geodesic is $\sqrt{g_{ij}(\vec{\tau}_B)\xi^i\xi^j}$. This definition differs from the one used here by a multiplicative factor of τ_{2B}. Since this transforms covariantly under a duality transformation, both choices of $\vec{\xi}$ would give manifestly covariant Feynman rules.

invariant under simultaneous transformation of $\vec{\tau}_B$, $\vec{J}$ or $\vec{\psi}$ and of course the charges Q_e and Q_m. In particular the 1PI vacuum amplitude $\widetilde{\Gamma}_B(\vec{\tau}, \vec{0})$ will be duality invariant under the transformation (4.29).

We shall now give an algorithm for explicitly generating duality covariant vertices in this 0-dimensional field theory. For this we need to expand the duality invariant function $F(\vec{\tau})$ in a Taylor series expansion in $\vec{\xi}$. This is given by:

$$F(\vec{\tau}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\tau_{B2})^n \xi_{i_1} \dots \xi_{i_n} \, D_{i_1} \cdots D_{i_n} F(\vec{\tau}) \Big|_{\vec{\tau} = \vec{\tau}_B}, \tag{F.18}$$

where D_i denotes covariant derivative with respect to τ_i, computed using the affine connection Γ^i_{jk} constructed from the metric (F.13). We arrive at (F.18) by using the result that in the $\vec{\xi}$ coordinate system symmetrized covariant derivatives are equal to ordinary derivatives. Using this we can replace ordinary derivatives in the Taylor series expansion to covariant derivatives with respect to ξ_i. In the second step we use the tensorial transformation properties of covariant derivatives to convert covariant derivative with respect to ξ_i to covariant derivative with respect to τ_i. The $(\tau_{B2})^n$ factor in (F.18) arises due to the fact that near $\vec{\tau} = \vec{\tau}_B$,

$$d\tau_i = \tau_{B2} d\xi_i. \tag{F.19}$$

In the $(\tau, \bar{\tau})$ coordinate system the non-zero components of the connection are

$$\Gamma^\tau_{\tau\tau} = \frac{i}{\tau_2}, \qquad \Gamma^{\bar{\tau}}_{\bar{\tau}\bar{\tau}} = -\frac{i}{\tau_2}. \tag{F.20}$$

The curvature tensor computed from this connection has the form

$$R^i_{jkl} = -(\delta^i_k g_{jl} - \delta^i_l g_{jk}), \tag{F.21}$$

which shows that the metric (F.13) describes a constant negative curvature metric. From (F.20) it follows that

$$D_\tau(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) = (\partial_\tau - im/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})),$$
$$D_{\bar{\tau}}(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})) = (\partial_{\bar{\tau}} + in/\tau_2)(D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})), \tag{F.22}$$

for any arbitrary ordering of D_τ and $D_{\bar{\tau}}$ in $D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau})$. (F.22) provides us with explicit expressions for the covariant derivatives of F appearing in (F.18). Also using (F.22) one can prove iteratively that under a duality transformation

$$(\tau_2)^{m+n} D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau}) \to \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{m-n} (\tau_2)^{m+n} D_\tau^m D_{\bar{\tau}}^n F(\vec{\tau}), \tag{F.23}$$

under a duality transformation. This establishes manifest duality covariance of the vertices constructed from (F.18).

We also need to worry about the contribution from the integration measure. The original measure $d^2\tau/(\tau_2)^2$ was duality invariant. Since duality transformation acts on $\vec{\xi}$ as a rotation, $d^2\xi$ is also a duality invariant measure. Thus we must have

$$\frac{d^2\tau}{(\tau_2)^2} = \mathcal{J}(\tau_B, \vec{\xi}) \, d^2\xi, \tag{F.24}$$

for some duality invariant function $\mathcal{J}(\tau_B, \vec{\xi})$. It has been shown in appendix G that

$$\mathcal{J}(\tau_B, \vec{\xi}) = \frac{1}{|\vec{\xi}|} \sinh |\vec{\xi}|. \tag{F.25}$$

We can now regard $-\ln \mathcal{J}(\tau_B, \vec{\xi})$ as an additional contribution to the action and expand this in a power series expansion in $\vec{\xi}$ to generate additional vertices. Using the expression for $F(\vec{\tau})$ given in (4.28) we now see that the full 'action' is given by

$$
\begin{aligned}
F(\vec{\tau}) - \ln \mathcal{J}(\tau_B, \vec{\xi}) = -\Bigg[& \frac{\pi}{2\tau_2} |Q_e + \tau Q_m|^2 - \ln f^{(k)}(\tau) - \ln f^{(k)}(-\bar{\tau}) - (k+2)\ln(2\tau_2) \\
& + \ln \left\{ K_0 \frac{\pi}{\tau_2} |Q_e + \tau Q_m|^2 \right\} + \ln \mathcal{J}(\tau_B, \vec{\xi}) \\
& + \ln \left(1 + \frac{2(k+3)\tau_2}{\pi|Q_e + \tau Q_m|^2} \right) \Bigg].
\end{aligned}
\tag{F.26}
$$

In this expression the first term inside the square bracket is quadratic in the charges, the last term contains terms of order Q^{-2n} for $n \geq 1$, and the other terms are of order Q^0. Thus in order to carry out a systematic expansion in powers of inverse charges we need to regard the first term as the tree level contribution, the last term as two and higher loop contributions and the other terms as the 1-loop contribution. We can now evaluate the effective action $\widetilde{\Gamma}_B(\vec{\tau}_B)$ in a systematic loop expansion. The leading term in the effective action is then just the first term in (F.26) evaluated at $\vec{\tau} = \vec{\tau}_B$:

$$\widetilde{\Gamma}_0(\tau_B) = -\frac{\pi}{2\tau_{2B}} |Q_e + \tau_B Q_m|^2. \tag{F.27}$$

At the next order we need to include the tree level contribution from the rest of the terms in the action (except the last term which is higher order) and one loop contribution from the first term. The former corresponds to these terms being evaluated at $\vec{\tau} = \vec{\tau}_B$, i.e. $\vec{\xi} = 0$. Since $\mathcal{J}(\vec{\tau}_B, \vec{\xi} = 0) = 1$, we get this contribution to be

$$\ln f^{(k)}(\tau_B) + \ln f^{(k)}(-\bar{\tau}_B) + (k+2)\ln(2\tau_{2B}) - \ln \left\{ K_0 \frac{\pi}{\tau_{2B}} |Q_e + \tau_B Q_m|^2) \right\}. \tag{F.28}$$

For the one loop contribution from the first term in the action we need to expand this term to quadratic order in $\vec{\xi}$ using eqs. (F.18), (F.22). The order $\vec{\xi}$ and ξ^2 terms are given by

$$-\frac{i\pi}{4\tau_{2B}} \left\{ \bar{\xi}(Q_e + Q_m \tau_B)^2 + \xi(Q_e + Q_m \bar{\tau}_B)^2 \right\} - \frac{\pi}{4\tau_{2B}} |Q_e + \tau_B Q_m|^2 \, \bar{\xi}\xi. \tag{F.29}$$

The linear term in $\vec{\xi}$ do not give any contribution to the 1PI amplitudes. The contribution from the quadratic term gives

$$\ln \left(\frac{1}{4\tau_{2B}} |Q_e + \tau_B Q_m|^2 \right). \tag{F.30}$$

Thus the complete one loop contribution to the effective action is given by

$$\widetilde{\Gamma}_1(\tau_B) = \ln f^{(k)}(\tau_B) + \ln f^{(k)}(-\bar{\tau}_B) + (k+2)\ln(2\tau_{2B}) - \ln(4\pi K_0). \tag{F.31}$$

Up to an additive constant this agrees with the black hole entropy function for CHL models calculated in [27].

G. The integration measure $\mathcal{J}(\vec{\tau}_B, \vec{\xi})$

In this appendix we shall compute the integration measure $\mathcal{J}(\vec{\tau}_B, \vec{\xi})$ which arises from a change of variables from τ_1, τ_2 to the normal coordinates:

$$\frac{d^2\tau}{(\tau_2)^2} = \mathcal{J}(\vec{\tau}_B, \vec{\xi})\, d^2\xi. \tag{G.1}$$

We first note that the duality invariant metric

$$\frac{1}{\tau_2^2}(d\tau_1^2 + d\tau_2^2) \tag{G.2}$$

describes a metric of constant negative curvature -1. Since this is a homogeneous space, $\mathcal{J}(\vec{\tau}_B, \vec{\xi})$ cannot depend on $\vec{\tau}_B$. Furthermore with an appropriate change of variables we can bring the metric and measure to the standard form:

$$ds^2 = d\theta^2 + \sinh^2\theta d\phi^2\,, \qquad \frac{d^2\tau}{\tau_2^2} = \sinh\theta\, d\theta\, d\phi\,. \tag{G.3}$$

Since $\mathcal{J}$ is independent of the base point $\vec{\tau}_B$, we can calculate it by taking the base point to be at $\theta = 0$. The geodesics passing through this point are constant ϕ lines, and the length measured along such a geodesic is given by θ. Thus we have

$$\vec{\xi} = (\theta\cos\phi, \theta\sin\phi)\,. \tag{G.4}$$

This gives

$$d^2\xi = \theta d\theta d\phi\,. \tag{G.5}$$

Comparing this with (G.3) we get

$$\frac{d^2\tau}{\tau_2^2} = \frac{\sinh\theta}{\theta}d^2\xi = \frac{1}{|\vec{\xi}|}\sinh|\vec{\xi}|\, d^2\xi\,. \tag{G.6}$$

Thus

$$\mathcal{J}(\vec{\tau}_B, \vec{\xi}) = \frac{1}{|\vec{\xi}|}\sinh|\vec{\xi}|\,. \tag{G.7}$$

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